Consider the Einstein energy equation of special relativity:

\[ E^2 = p^2 c^2 + m^2 c^4 \]  \hspace{1cm} (1)

where \( E \) is the total relativistic energy, \( p \) is the relativistic momentum, \( m \) is the particle mass, and \( c \) is the speed of light in vacuo.

\( E \) is quantized with the same rules as in non-relativistic quantum mechanics:

\[ E_n = \hbar \omega_n \]  \hspace{1cm} (2)

\[ p_n = -\hbar \frac{\partial}{\partial \phi} \]  \hspace{1cm} (3)

The total wave function \( \psi \) can be reduced to the Schrödinger equation as follows:

\[ (E - mc^2)(E + mc^2) = p^2 c^2 \]  \hspace{1cm} (4)

so

\[ E - mc^2 = \frac{p^2 c^2}{E + mc^2} \]  \hspace{1cm} (5)

In the equation:

\[ H = E + U \]  \hspace{1cm} (6)

where \( U \) is the potential energy, so:

\[ H - U = mc^2 = \frac{p^2 c^2}{E + mc^2} \]  \hspace{1cm} (7)

Define:

\[ H_0 = H - mc^2 \]  \hspace{1cm} (8)

so

\[ H_0 = \frac{p^2 c^2}{E + mc^2} + U \]  \hspace{1cm} (9)

In general:

\[ E = \gamma mc^2 \]  \hspace{1cm} (10)
\[ Y = \left( 1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}} \]  \hspace{1cm} (11)

is the Lorentz factor.

In the limit:

\[ v \ll c \]  \hspace{1cm} (12)

it follows that

\[ Y \rightarrow 1 \]  \hspace{1cm} (13)

and

\[ H_0 = \frac{p^2}{2mc^2} + U \]  \hspace{1cm} (14)

i.e.

\[ H_0 = \frac{p^2}{2m} + U \]  \hspace{1cm} (15)

which is the classical result. \( \Box \ \text{E.D. Eq. (15)} \).

question: what does this mean?

\[ H_0 \psi = -\frac{\nabla^2 \psi}{2m} + U_0 \psi \]  \hspace{1cm} (16)

This is the Schrodinger equation:

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + U_0 \psi = i \hbar \frac{\partial \psi}{\partial t} \]  \hspace{1cm} (17)

If

\[ \psi = \psi_1 \psi_2 \]  \hspace{1cm} (18)

Eq. (17) splits into:

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_1}{\partial x^2} + U(x) \psi_1 = E \psi_1 \]  \hspace{1cm} (19)

and

\[ i \hbar \frac{\partial \psi_2}{\partial t} = -E \psi_2 \]  \hspace{1cm} (20)
(3) The solution of eq. (20) is:

$$\psi_2 = \exp \left( -i \frac{Et}{\hbar} \right)$$  \hspace{1cm} (a1)

In the non-relativistic Schrödinger limit:

$$E_n = -\frac{\mu e^2}{32 \pi^2 \epsilon_0^2 \hbar^2} \frac{n^2}{n^2}$$  \hspace{1cm} (a2)

where \(\mu\) is the reduced mass, \(\epsilon_0\) is the permittivity of the vacuum, \(e\) is the charge on the proton, \(\hbar\) is the reduced Planck constant, and \(n\) is the principal quantum number.

These results have been obtained by using the non-relativistic approximation (12). The rigorously correct non-relativistic approximations are:

$$E_{nJ} = E_n \left( 1 + \left( \frac{\alpha}{n} \right)^2 \left( \frac{\hbar}{J^{1/2}} - \frac{3}{4} \right) \right)$$  \hspace{1cm} (a3)

where \(\alpha\) is the fine-structure constant and \(J\) is the total angular momentum quantum number:

$$J = L + S, \ldots, L - S - (24)$$

and

$$L = 0, \ldots, n-1$$

The spin quantum number takes \(\pm \frac{1}{2}\) values:

$$S = \pm \frac{1}{2}$$  \hspace{1cm} (a6)

The quantum numbers \(J\) and \(S\) are missing from the Schrödinger H atom because of the above derivation of the Schwinger expansion for the \(\text{SU}(2)\) basis.

The energy equation (1) does not use the energy levels of \(2p_{1/2}\) and \(2p_{3/2}\) in the Dirac H atom.
and \( S_{1/2} \) are 0 if some because \( J = 0 \) some 0.

2 S\(_{1/2} \): \( n = 1, L = 0, S = 1/2, J = 1/2 \) \( (27) \)

2 P\(_{1/2} \): \( n = 1, L = 1, S = -1/2, J = 1/2 \) \( (28) \)

Therefore there is no Landshoff in the Dirac Hamiltonian.