

```
(%i1) kill(all);
(%o0) done

(%i1) assume(c>0, m>0);
(%o1) [c>0, m>0]
```

## Non-relativistic theory

### 1 Equations of central motion in coordinates (r,phi)

```
(%i2) depends([r,phi],t);
(%o2) [r(t), phi(t)]

(%i3) diff(r,t,2) = diff(phi,t)^2*r - M*G/r^2;
(%o3)  $\frac{d^2}{dt^2} r = \left(\frac{d}{dt} \phi\right)^2 r - \frac{GM}{r^2}$ 

(%i4) diff(phi,t,2) = -2*diff(phi,t)*diff(r,t)/r;
(%o4)  $\frac{d^2}{dt^2} \phi = -\frac{2\left(\frac{d}{dt} \phi\right)\left(\frac{d}{dt} r\right)}{r}$ 
```

### 2 Non-relativistic Hamilton equations I inertial frame $v^2 = p_r^2/m$

```
(%i5) H: (p_r^2)/(2*m) - m*M*G/q_r;
(H)  $\frac{p_r^2}{2m} - \frac{GMm}{q_r}$ 
```

#### 2.1 First Hamilton equations

```
(%i6) H1: q_rd = diff(H,p_r);
(H1)  $q_{rd} = \frac{p_r}{m}$ 

(%i7) H2: q_phid = diff(H,p_phi);
(H2)  $q_{phid} = 0$ 
```

#### 2.2 Second Hamilton equations

```
(%i8) H3: p_rd = -diff(H,q_r);
(H3)  $p_{rd} = -\frac{GMm}{q_r^2}$ 

(%i9) H4: p_phid = -diff(H,q_phi);
(H4)  $p_{phid} = 0$ 
```

### 3 Non-relativistic Hamilton equations II general frame $p^2 = p_r^2 + p_phi^2 / q_r^2$

```
(%i10) H: (p_r^2 + 1/q_r^2*p_phi^2)/(2*m) - m*M*G/q_r;
(H)  $\frac{\frac{p_{phi}^2}{q_r^2} + p_r^2}{2m} - \frac{GMm}{q_r}$ 
```

#### 3.1 First Hamilton equations

```
(%i11) H1: q_rd = diff(H,p_r);
(H1)  $q_{rd} = \frac{p_r}{m}$ 
```

```
(%i12) H2: q_phid = diff(H,p_phi);
```

$$(H2) \quad q_{phid} = \frac{p_{phi}}{m q_r^2}$$

### 3.2 Second Hamilton equations

```
(%i13) H3: p_rd = -diff(H,q_r);
```

$$(H3) \quad p_{rd} = \frac{p_{phi}^2}{m q_r^3} - \frac{G M m}{q_r^2}$$

```
(%i14) H4: p_phid = -diff(H,q_phi);
```

$$(H4) \quad p_{phid} = 0$$

## Relativistic theory

### 1 Relativistic Hamilton equations I

#### inertial frame

#### gamma defined by velocity $q_{rd}^2$

```
(%i15) gamma: (1-(q_rd^2)/c^2)^(-1/2);
```

$$(gamma) \quad \frac{1}{\sqrt{1 - \frac{q_{rd}^2}{c^2}}}$$

```
(%i16) H: 1/(2*gamma)*((p_r^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;
```

$$(H) \quad \frac{\left(\frac{p_r^2}{m} + c^2 m\right) \sqrt{1 - \frac{q_{rd}^2}{c^2}}}{2} - \frac{G M m}{q_r}$$

#### 1.1 First Hamilton equations

```
(%i17) H1: q_rd = diff(H,p_r);
```

$$(H1) \quad q_{rd} = \frac{p_r \sqrt{1 - \frac{q_{rd}^2}{c^2}}}{m}$$

```
(%i18) H2: q_phid = diff(H,p_phi);
```

$$(H2) \quad q_{phid} = 0$$

#### 1.2 Second Hamilton equations

```
(%i19) H3: p_rd = (-diff(H,q_r));
```

$$(H3) \quad p_{rd} = -\frac{G M m}{q_r^2}$$

```
(%i20) H4: p_phid = -diff(H,q_phi);
```

$$(H4) \quad p_{phid} = 0$$

#### 1.3 Re-insert gamma

```
(%i24) ratsubst(%gamma, gamma, H1);
ratsubst(%gamma, gamma, H2);
expand(ratsubst(%gamma, gamma, H3));
ratsubst(%gamma, gamma, H4);
```

$$(%o21) \quad q_{rd} = \frac{p_r}{\gamma m}$$

$$(%o22) \quad q_{phid} = 0$$

$$(%o23) \quad p_{rd} = -\frac{G M m}{q_r^2}$$

$$(%o24) \quad p_{phid} = 0$$

## 2 Relativistic Hamilton equations II inertial frame gamma defined by velocity $p_r^2/m^2$

(%i25)  $\text{gamma} := (1 - (p_r^2) / (m^2 * c^2))^{(-1/2)}$ ;

(gamma) 
$$\frac{1}{\sqrt{1 - \frac{p_r^2}{c^2 m^2}}}$$

(%i26)  $H := 1 / (1 * \text{gamma}) * ((p_r^2) * c^2 / (m * c^2) + m * c^2) - m * M * G / q_r$ ;

(H) 
$$\sqrt{1 - \frac{p_r^2}{c^2 m^2}} \left( \frac{p_r^2}{m} + c^2 m \right) - \frac{G M m}{q_r}$$

### 2.1 First Hamilton equations

(%i27)  $H1 := q_{rd} = \text{diff}(H, p_r)$ ;

(H1) 
$$q_{rd} = \frac{2 p_r \sqrt{1 - \frac{p_r^2}{c^2 m^2}}}{m} - \frac{p_r \left( \frac{p_r^2}{m} + c^2 m \right)}{c^2 m^2 \sqrt{1 - \frac{p_r^2}{c^2 m^2}}}$$

(%i28)  $H2 := q_{phid} = \text{diff}(H, p_{phi})$ ;

(H2)  $q_{phid} = 0$

### 2.2 Second Hamilton equations

(%i29)  $H3 := p_{rd} = (-\text{diff}(H, q_r))$ ;

(H3) 
$$p_{rd} = -\frac{G M m}{q_r^2}$$

(%i30)  $H4 := p_{phid} = -\text{diff}(H, q_{phi})$ ;

(H4)  $p_{phid} = 0$

### 2.3 Re-insert gamma

(%i34)  $\text{expand}(\text{ratsubst}(\%gamma, \text{gamma}, H1))$ ;  
 $\text{ratsubst}(\%gamma, \text{gamma}, H2)$ ;  
 $\text{ratsubst}(\%gamma, \text{gamma}, H3)$ ;  
 $\text{ratsubst}(\%gamma, \text{gamma}, H4)$ ;

(%o31) 
$$q_{rd} = -\frac{\gamma p_r^3}{c^2 m^3} - \frac{\gamma p_r}{m} + \frac{2 p_r}{\gamma m}$$

(%o32)  $q_{phid} = 0$

(%o33) 
$$p_{rd} = -\frac{G M m}{q_r^2}$$

(%o34)  $p_{phid} = 0$

## 3 Relativistic Hamilton equations IV general frame gamma with $q_{rd}^2 + q_{phid}^2 * q_r^2$

(%i35)  $\text{gamma} := (1 - (q_{rd}^2 + q_r^2 * q_{phid}^2) / (c^2))^{(-1/2)}$ ;

(gamma) 
$$\frac{1}{\sqrt{1 - \frac{q_{rd}^2 + q_{phid}^2 q_r^2}{c^2}}}$$

(%i36)  $H := 1 / (1 * \text{gamma}) * ((p_r^2 + p_{phi}^2 / q_r^2) * c^2 / (m * c^2) + m * c^2) - m * M * G / q_r$ ;

(H) 
$$\left( \frac{p_{phi}^2}{q_r^2} + p_r^2 \right) \sqrt{1 - \frac{q_{rd}^2 + q_{phid}^2 q_r^2}{c^2}} - \frac{G M m}{q_r}$$

### 3.1 First Hamilton equations

```
(%i37) H1: q_rd = diff(H,p_r);
```

$$(H1) \quad q_{rd} = \frac{2 p_r \sqrt{1 - \frac{q_{rd}^2 + q_{phid}^2 q_r^2}{c^2}}}{m}$$

```
(%i38) H2: q_phid = diff(H,p_phi);
```

$$(H2) \quad q_{phid} = \frac{2 p_{phi} \sqrt{1 - \frac{q_{rd}^2 + q_{phid}^2 q_r^2}{c^2}}}{m q_r^2}$$

### 3.2 Second Hamilton equations

```
(%i39) H3: p_rd = -diff(H,q_r);
```

$$(H3) \quad p_{rd} = \frac{2 p_{phi}^2 \sqrt{1 - \frac{q_{rd}^2 + q_{phid}^2 q_r^2}{c^2}}}{m q_r^3} + \frac{q_{phid}^2 \left( \frac{p_{phi}^2}{q_r^2} + p_r^2 \right)}{m + c^2 m} q_r - \frac{G M m}{q_r^2}$$

```
(%i40) H4: p_phid = -diff(H,q_phi);
```

$$(H4) \quad p_{phid} = 0$$

### 3.3 Re-insert gamma

```
(%i44) Hr1: ratsubst(%gamma, gamma, H1);
Hr2: ratsubst(%gamma, gamma, H2);
Hr3: expand(ratsubst(%gamma, gamma, H3));
Hr4: ratsubst(%gamma, gamma, H4);
```

$$(Hr1) \quad q_{rd} = \frac{2 p_r}{\gamma m}$$

$$(Hr2) \quad q_{phid} = \frac{2 p_{phi}}{\gamma m q_r^2}$$

$$(Hr3) \quad p_{rd} = \frac{\gamma p_r^2 q_{phid}^2 q_r}{c^2 m} + \gamma m q_{phid}^2 q_r + \frac{\gamma p_{phi}^2 q_{phid}^2}{c^2 m q_r} - \frac{G M m}{q_r^2} + \frac{2 p_{phi}^2}{\gamma m q_r^3}$$

$$(Hr4) \quad p_{phid} = 0$$

```
(%i45) Hr: expand((first((solve([Hr1,Hr2,Hr3,Hr4],
[q_rd, q_phid, p_rd, p_phid])))))$
```

```
(%i49) Hr[1]; Hr[2]; Hr[3]; Hr[4];
```

$$(%o46) \quad q_{rd} = \frac{2 p_r}{\gamma m}$$

$$(%o47) \quad q_{phid} = \frac{2 p_{phi}}{\gamma m q_r^2}$$

$$(%o48) \quad p_{rd} = -\frac{G M m}{q_r^2} + \frac{4 p_{phi}^2 p_r^2}{\gamma c^2 m^3 q_r^3} + \frac{6 p_{phi}^2}{\gamma m q_r^3} + \frac{4 p_{phi}^4}{\gamma c^2 m^3 q_r^5}$$

$$(%o49) \quad p_{phid} = 0$$

Check

```
(%i50) s: expand(ratsubst(rhs(Hr2), q_phid, Hr3));
```

$$(s) \quad p_{rd} = -\frac{G M m}{q_r^2} + \frac{4 p_{phi}^2 p_r^2}{\gamma c^2 m^3 q_r^3} + \frac{6 p_{phi}^2}{\gamma m q_r^3} + \frac{4 p_{phi}^4}{\gamma c^2 m^3 q_r^5}$$

```
(%i51) s3: p_rd=- (G*M*m)/q_r^2+(6*p_phi^2)/(%gamma*m*q_r^3)
+4*p_phi^2/(%gamma*c^2*m^3*q_r^3)*(p_r^2+p_phi^2/q_r^2);
```

$$(s3) \quad p_{rd} = -\frac{G M m}{q_r^2} + \frac{4 p_{\phi}^2 \left( \frac{p_{\phi}^2}{q_r^2} + p_r^2 \right)}{\gamma c^2 m^3 q_r^3} + \frac{6 p_{\phi}^2}{\gamma m q_r^3}$$

```
(%i53) ratsimp(Hr[3]-s);
ratsimp(Hr[3]-s3);
```

```
(%o52) 0=0
```

```
(%o53) 0=0
```

#### 4 Relativistic Hamilton equations IV

**general frame**

**gamma defined by velocities  $(p_r^2 + p_{\phi}^2/q_r^2)/m^2$**

```
(%i54) gamma: (1-(p_r^2+1/q_r^2*p_phi^2)/(m^2*c^2))^(1/2);
```

$$(gamma) \quad \frac{1}{\sqrt{1 - \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{c^2 m^2}}}$$

```
(%i55) H: 1/(1*gamma)*((p_r^2+p_phi^2/q_r^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;
```

$$(H) \quad \sqrt{1 - \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{c^2 m^2}} \left( \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{m} + c^2 m \right) - \frac{G M m}{q_r}$$

##### 4.1 First Hamilton equations

```
(%i56) H1: q_rd = diff(H,p_r);
```

$$(H1) \quad q_{rd} = \frac{2 p_r \sqrt{1 - \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{c^2 m^2}}}{m} - \frac{p_r \left( \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{m} + c^2 m \right)}{c^2 m^2 \sqrt{1 - \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{c^2 m^2}}}$$

```
(%i57) H2: q_phid = diff(H,p_phi);
```

$$(H2) \quad q_{phid} = \frac{2 p_{\phi} \sqrt{1 - \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{c^2 m^2}}}{m q_r^2} - \frac{p_{\phi} \left( \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{m} + c^2 m \right)}{c^2 m^2 \sqrt{1 - \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{c^2 m^2}} q_r^2}$$

##### 4.2 Second Hamilton equations

```
(%i58) H3: p_rd = (-diff(H,q_r));
```

$$(H3) \quad p_{rd} = -\frac{G M m}{q_r^2} - \frac{p_{\phi}^2 \left( \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{m} + c^2 m \right)}{c^2 m^2 \sqrt{1 - \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{c^2 m^2}} q_r^3} + \frac{2 p_{\phi}^2 \sqrt{1 - \frac{\frac{p_{\phi}^2}{q_r^2} + p_r^2}{c^2 m^2}}}{m q_r^3}$$

```
(%i59) H4: p_phid = -diff(H,q_phi);
```

```
(H4) p_phid=0
```

### 4.3 Re-insert gamma

```
(%i63) Hr1: expand(ratsubst(%gamma, gamma, H1));
      Hr2: expand(ratsubst(%gamma, gamma, H2));
      Hr3: expand(ratsubst(%gamma, gamma, H3));
      Hr4: ratsubst(%gamma, gamma, H4);
```

$$(Hr1) \quad q_{rd} = -\frac{\gamma p_{phi}^2 p_r}{c^2 m^3 q_r^2} - \frac{\gamma p_r^3}{c^2 m^3} - \frac{\gamma p_r}{m} + \frac{2 p_r}{\gamma m}$$

$$(Hr2) \quad q_{phid} = -\frac{\gamma p_{phi} p_r^2}{c^2 m^3 q_r^2} - \frac{\gamma p_{phi}}{m q_r^2} + \frac{2 p_{phi}}{\gamma m q_r^2} - \frac{\gamma p_{phi}^3}{c^2 m^3 q_r^4}$$

$$(Hr3) \quad p_{rd} = -\frac{G M m}{q_r^2} - \frac{\gamma p_{phi}^2 p_r^2}{c^2 m^3 q_r^3} - \frac{\gamma p_{phi}^2}{m q_r^3} + \frac{2 p_{phi}^2}{\gamma m q_r^3} - \frac{\gamma p_{phi}^4}{c^2 m^3 q_r^5}$$

$$(Hr4) \quad p_{phid} = 0$$

### 4.4 Simplify by hand

```
(%i66) Hh1: q_rd = -%gamma*p_r/(c^2*m^3)*(p_phi^2/q_r^2+p_r^2)
      +p_r/m*(2/%gamma-%gamma);
      Hh2: q_phid=-%gamma*p_phi/(c^2*m^3*q_r^2)*(p_r^2+p_phi^2/q_r^2)
      +p_phi/(m*q_r^2)*(2/%gamma-%gamma);
      Hh3: p_rd=-%gamma*p_phi^2/(c^2*m^3*q_r^3)*(p_r^2+p_phi^2/q_r^2)
      +p_phi^2/(m*q_r^3)*(2/%gamma-%gamma)-(G*M*m)/q_r^2
      ;
```

$$(Hh1) \quad q_{rd} = \frac{\left(\frac{2}{\gamma} - \gamma\right) p_r}{m} - \frac{\gamma p_r \left(\frac{p_{phi}^2}{q_r^2} + p_r^2\right)}{c^2 m^3}$$

$$(Hh2) \quad q_{phid} = \frac{\left(\frac{2}{\gamma} - \gamma\right) p_{phi}}{m q_r^2} - \frac{\gamma p_{phi} \left(\frac{p_{phi}^2}{q_r^2} + p_r^2\right)}{c^2 m^3 q_r^2}$$

$$(Hh3) \quad p_{rd} = -\frac{G M m}{q_r^2} - \frac{\gamma p_{phi}^2 \left(\frac{p_{phi}^2}{q_r^2} + p_r^2\right)}{c^2 m^3 q_r^3} + \frac{\left(\frac{2}{\gamma} - \gamma\right) p_{phi}^2}{m q_r^3}$$

```
(%i69) ratsimp(Hh1-Hr1);
      ratsimp(Hh2-Hr2);
      ratsimp(Hh3-Hr3);
```

$$(%o67) \quad 0 = 0$$

$$(%o68) \quad 0 = 0$$

$$(%o69) \quad 0 = 0$$