[...]26(3): Brief Review of the Hamilton Jacobi Dynamics

The essence of the method is to transform the Hamilton dynamics to:

$$\dot{Q} = \frac{\partial H}{\partial P} = 0 \quad -(1)$$

and

$$\dot{P} = -\frac{\partial H}{\partial Q} = 0 \quad -(2)$$

using the Hamilton Jacobi equation:

$$H + \frac{\partial S}{\partial t} = 0 \quad -(3)$$

where the generating function S is the action:

$$S = \int \mathcal{L} \, dt \quad -(4)$$

and where $\mathcal{L}$ is the Lagrangian.

The original Hamilton equations are:

$$\dot{q} = \frac{\partial H}{\partial p} \quad -(5)$$

and

$$\dot{p} = -\frac{\partial H}{\partial q} \quad -(6)$$

Hamilton's principal function is defined by:

$$S(q, t) = \int^{(q, t)} \mathcal{L} \, dt \quad -(7)$$

When the Hamiltonian does not depend on time:

$$S = W - Et \quad -(8)$$

where W is Hamilton's characteristic function. The reduced Hamilton Jacobi equation is then:

$$H\left(q, \frac{\partial S}{\partial q}\right) = E \quad -(9)$$

The most useful feature of the Hamilton Jacobi equation is the definition of constants of motion. This

2) uses the idea that S is completely separable:

$$S = S_1(q_1) + S_2(q_2) + \dots + S_N(q_N) - Et \quad -(10)$$

The same idea is used in the Schroedinger equation where the wave function is expressed as a product:

$$\phi = \phi_t \, \phi_r \, \phi_v \dots \quad -(11)$$

In the Hamilton Jacobi dynamics:

$$p = \frac{\partial S}{\partial q} \quad -(12)$$

and this may be extended to:

$$p^\mu = \frac{\partial S}{\partial x^\mu} \quad -(13)$$

The Maxima integrator can be used to integrate the equations that emerge from the Hamilton Jacobi analysis, so there is no longer any difficulty with the approach.

For example if we consider motion in a plane described by:

$$H = \frac{1}{2m}\left(p_x^2 + p_z^2\right) + mgz \quad -(14)$$

the Hamilton Jacobi equation is:

$$\frac{1}{2m}\left(\left(\frac{\partial S(x,z,t)}{\partial x}\right)^2 + \left(\frac{\partial S(x,z,t)}{\partial z}\right)^2\right) + mgz + \frac{\partial S}{\partial t}(x,z,t) = 0 \quad -(15)$$

with the following separation of variables:

$$S(x,z,t) = W(x) + W(z) - Et \quad -(16)$$

3) A function depending on x can only equal a function independent of x if both are constants. Similarly for z. The constants are α_x and α_z and are constants of motion.

Therefore:

$$\frac{1}{2m}\left(\frac{dW_x(x)}{dx}\right)^2 = \alpha_x \quad -(17)$$

and

$$\frac{1}{2m}\left(\frac{dW_z(z)}{dz}\right)^2 + mgz = \alpha_z \quad -(18)$$

These equations happen to be solvable analytically:

$$W_x(x) = \pm x\,(2m\alpha_x)^{1/2} \quad -(19)$$

$$W_z(z) = \pm\left(\frac{8}{9mg^2}\right)\left(\alpha_z - mgz\right)^{3/2} \quad -(20)$$

The other constants of motion are β_x and β_z, defined by:

$$\beta_x = \frac{\partial S}{\partial \alpha_x} = \pm x\left(\frac{m}{2\alpha_x}\right)^{1/2} - t \quad -(21)$$

and

$$\beta_z = \frac{\partial S}{\partial \alpha_z} = \pm\left(\frac{2(\alpha_z - mgz)}{mg^2}\right)^{1/2} - t \quad -(22)$$

so the trajectory is:

$$x = \pm\left(\frac{2\alpha_x}{m}\right)(\beta_x + t) \quad -(23)$$

and

$$z = \frac{\alpha_z}{mg} - \frac{g}{2}(\beta_z + t)^2 \quad -(24)$$

†) Motion in orbital dynamics is described in plane polar coordinates by:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{mMG}{r} \quad -(25)$$

where

$$U = -\frac{mMG}{r} \quad -(26)$$

Attraction between an electron and proton is described by

$$U = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(27)$$

The Hamilton Jacobi equation is:

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial r}\right)^2 + U(r) + \frac{1}{2mr^2}\left(\frac{\partial S_0}{\partial \phi}\right)^2 = E \quad -(28)$$

where

$$S_0(r, \phi) = S_r(r) + S_\phi(\phi) \quad -(29)$$

Here,

$$\left(\frac{\partial S_\phi}{\partial \phi}\right)^2 = \beta = L^2 \quad -(30)$$

is a constant of motion, where L is the angular momentum. So

$$L = \frac{\partial S_\phi}{\partial \phi} = \text{constant} \quad -(31)$$

i.e.

$$\frac{dL}{dt} = 0 \quad -(32)$$

The original Hamilton equation is:

$$L = \frac{\partial H}{\partial \phi} \quad -(33)$$

and does not show that L is a constant of motion without a Lagrangian analysis.

Therefore eq. (28) reduces to:

$$\frac{1}{2m}\left(\frac{\partial S_r}{\partial r}\right)^2 + U(r) + \frac{L^2}{2mr^2} = E \quad -(34)$$

where E is the total energy, a constant of motion:

$$\frac{dE}{dt} = 0 \quad -(35)$$

In eq. (34):

$$p_r = \frac{\partial S_r}{\partial r} \quad -(36)$$

Eq. (34) in orbital theory is:

$$\frac{1}{2m}\left(\frac{\partial S_r}{\partial r}\right)^2 - \frac{mMG}{r} + \frac{L^2}{2mr^2} = E \quad -(37)$$

and may be solved for S_r, p_r and:

$$q_r = \frac{\partial S}{\partial p_r} \quad -(38)$$

The equivalent of eq. (34) in quantum mechanics is:

$$\frac{d^2 P}{dr^2} + \left(\frac{2m}{\hbar^2}\right)\left(\frac{e^2}{4\pi\epsilon_0 r} - \frac{l(l+1)\hbar^2}{2mr^2}\right) P = -\frac{2mE}{\hbar^2} P \quad -(39)$$

which is equivalent to the Schroedinger equation.

The most powerful dynamics are however the Evans Eckardt dynamics:

$$\frac{dH}{dt} = 0 \quad -(40)$$

$$\frac{dL}{dt} = 0 \quad -(41)$$