48(3): Hamiltonian Method and Smooth Transition to the Classical Limit

Consider firstly the hamiltonian of flat space time:

$$H = \gamma m c^2 - \frac{mMG}{r} \quad -(1)$$

$$= T + U + E_0$$

where T is the kinetic energy, U is the potential energy and E_0 the rest energy

$$E_0 = mc^2 \quad -(2)$$

The smooth transition to classical theory is achieved with

$$H_0 = H - mc^2 = T + U \quad -(3)$$

$$= (\gamma - 1) mc^2 - \frac{mMG}{r} \quad -(4)$$

H_0 is a conserved constant of motion, so:

$$\frac{dH_0}{dt} = 0 \quad -(5)$$

Write:

$$\frac{dH_0}{dt} = \frac{dH_0}{dr}\frac{dr}{dt} \quad -(6)$$

It follows that:

$$\frac{dH_0}{dr} = 0 \quad -(7)$$

i.e.

$$F = \frac{d}{dr}\left((\gamma - 1) mc^2\right) = -\frac{mMG}{r^2} \quad -(8)$$

which is the force equation derived from the hamiltonian (4).

Note that:

$$T = (\gamma - 1) mc^2 \quad -(9)$$

is the relativistic kinetic energy. Therefore:

$$F = \frac{dT}{dr} \quad -(10)$$

which is the differential form of the work equation:

$$W_{12} = T_2 - T_1 = \int_1 \underline{F} \cdot \underline{dr} \quad -(11)$$

2) If we start from rest:

$$T_1 = 0. \quad -(12)$$

Consider the force of the spaceship:

$$\underline{F} = \frac{d\underline{p}}{dt} = \frac{d}{dt}(\gamma m \underline{v}) \quad -(13)$$

where

$$\underline{p} = \gamma m \underline{v} \quad -(14)$$

is the relativistic momentum of the spaceship. It follows that

$$T = m\int_0^v \frac{d}{dt}(\gamma m \underline{v}) \cdot \underline{v} \quad -(14)$$

$$= m\int_0^v v\, d(\gamma v)$$

This expression can be integrated by parts. Consider:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}. \quad -(15)$$

It follows that:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \quad -(16)$$

i.e

$$\int u\,dv = uv - \int v\,du \quad -(17)$$

So

$$T = \gamma m v^2 - m\int_0^v \gamma u\,du \quad -(18)$$

$$= \gamma m v^2 + mc^2\left(1 - \frac{v^2}{c^2}\right)\Big|_0^v$$

$$= \gamma m v^2 + mc^2\left(1 - \frac{v^2}{c^2}\right)^{1/2} - mc^2$$

and the famous

$$E_0 = mc^2 \quad -(19)$$

has appeared. Finally use:

3) $$v^2 = c^2\left(1 - \frac{1}{\gamma^2}\right) \quad -(20)$$

to find that:

$$T = \gamma mc^2\left(1 - \frac{1}{\gamma^2}\right) + \frac{mc^2}{\gamma} - mc^2$$

$$= (\gamma - 1)mc^2 \quad -(21)$$

Q.E.D.

It follows that eq. (8) is: $\quad -(22)$

$$F = \gamma m v = \frac{d}{dr}\left((\gamma-1)mc^2\right) = -\frac{mMG}{r^2}$$

The smooth transition to classical theory is achieved with:

$$T = (\gamma - 1)mc^2 \xrightarrow[v \ll c]{} \frac{1}{2}mv^2 \quad -(23)$$

using

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \sim 1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots \quad -(24)$$

Now consider the reduced hamiltonian of m theory:

$$H_0 = m(r_1)(\gamma - 1)mc^2 - \frac{mMG}{r_1} \quad -(25)$$

where

$$\gamma = \left(m(r) - \frac{v^2}{m(r)c^2}\right)^{-1/2} \quad -(26)$$

$$= \left(m(r_1) - \frac{\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1}{c^2}\right)^{-1/2}$$

and

$$r_1 = \frac{r}{m(r)^{1/2}} \quad -(27)$$

It follows that

$$\frac{dH_0}{dt} = \frac{dH_0}{dr_1}\frac{dr_1}{dt} = 0 \quad -(28)$$

and:

$$\frac{dH_0}{dr_1} = \frac{d}{dr_1}\left(n(r_1)(\gamma - 1) mc^2 \right) = -\frac{mMG}{r_1^2} \quad -(29)$$

$$= (\gamma-1) mc^2 \frac{dn(r_1)}{dr_1} + n(r_1)\frac{d}{dr_1}\left((\gamma-1)mc^2\right)$$

The first term on the right hand side of this equation is the reduced vacuum force, which reduces smoothly to its classical equivalent. The second term is the force:

$$F_1 = n(r_1)\frac{d}{dr_1}\left((\gamma-1)mc^2\right) \quad -(30)$$

where

$$T_1 = (\gamma-1)mc^2 \quad -(31)$$

is the relativistic kinetic energy of n theory:

$$T_1 = \int \underline{F_1} \cdot \underline{dr_1} \quad -(32)$$

and where:

$$\underline{F_1} = \frac{d}{dt}\left(\gamma m \underline{v_1}\right) \quad -(33)$$

with:

$$\underline{v_1} = \frac{v}{n(r)^{1/2}} \quad -(34)$$

It follows that:

$$T_1 = m\int_0^{v_1} \frac{d}{dt}\left(\gamma m \underline{v_1}\right) \cdot \underline{v_1}$$

$$= m\int_0^{v_1} v_1\, d(\gamma v_1) \quad -(35)$$

$$= (\gamma-1)mc^2$$

This is therefore a perfectly self consistent theory.

The equations of motion for this theory are:

$$\frac{dH_0}{dt} = 0 \quad -(36)$$

and

$$\frac{dL}{dt} = 0 \quad -(37)$$

which can be integrated numerically to give a vast amount of new information. In eq. (37):

$$L = \gamma m r_1^2 \dot{\phi} \quad -(38)$$

$$= \frac{\gamma m r^2 \dot{\phi}}{n(r)}$$

A smooth transition to the classical limit is achieved by using:

$$\gamma = \left(n(r_1) \left(1 - \frac{\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1}{n(r_1) c^2} \right) \right)^{-1/2} \quad -(39)$$

$$\sim n(r_1)^{1/2} \left(1 + \frac{1}{2} \frac{\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1}{n(r_1) c^2} + \cdots \right)$$

so

$$T_1 = (\gamma - 1) mc^2 \rightarrow \frac{m}{2} \frac{\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1}{n(r_1)} + mc^2 \left(n(r_1)^{1/2} - 1 \right) \quad -(40)$$

and therefore:

$$T_1 \xrightarrow[n(r) \to 1]{} \frac{1}{2} m v^2$$

Q.E.D.