417(7): The Role of $m(r)$ in the Classical Hamiltonian and Lagrangian.

Here The plane polar coordinate system of m space is (r_1, ϕ)

$$r_1 = \frac{r}{m(r)^{1/2}} \quad -(1)$$

so the Hamiltonian is:

$$H = m(r_1)\gamma m c^2 - \frac{mMG}{r_1} \quad -(2)$$

where

$$\gamma = \left(m(r_1) - \frac{\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1}{c^2}\right)^{-1/2} \quad -(3)$$

The Lagrangian is:

$$\mathcal{L} = -mc^2\gamma + \frac{mMG}{r_1} \quad -(4)$$

and the relativistic momentum is:

$$\underline{p}_1 = \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}_1} = \gamma m \dot{\underline{r}}_1. \quad -(5)$$

The Euler Lagrange equation is:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}_1} = \frac{\partial \mathcal{L}}{\partial \underline{r}_1} \quad -(6)$$

i.e.

$$\underline{F}_1 = m\frac{d}{dt}\left(\gamma \dot{\underline{r}}_1\right) = \frac{\partial \mathcal{L}}{\partial \underline{r}_1} \quad -(7)$$

In this equation:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}_1} = \underline{\nabla} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial r_1}\underline{e}_r \quad -(8)$$

In the (r, ϕ) coordinate system:

$$\mathcal{L} = -mc^2\left(m(r) - \frac{\dot{\underline{r}} \cdot \dot{\underline{r}}}{m(r)c^2}\right)^{1/2} + \frac{m(r)^{1/2} mMG}{r} \quad -(9)$$

As in Eq. (11), Note 417(1), eq. (5) is:

$$\frac{d\underline{p}_1}{dt} = \left(-\frac{mc^2}{2}\gamma \frac{dn(r)}{dr_1} - \frac{mMG}{r_1^2} \right) \underline{e}_r \quad -(10)$$

$$= \underline{F} + \underline{F}\,(\text{spacetime})$$

The spacetime force is:

$$\underline{F}\,(\text{spacetime}) = -\frac{mc^2}{2}\gamma \frac{dn(r)}{dr_1}\,\underline{e}_r \quad -(11)$$

i.e.

$$F(\text{spacetime}) = \frac{mc^2\,\gamma\, m(r)^{3/2}}{r\frac{dm(r)}{dr} - 2m(r)}\frac{dm(r)}{dr} \quad -(12)$$

in the (r, ϕ) coordinate system in which

$$\gamma = \left(m(r) - \frac{\dot{\underline{r}}\cdot\dot{\underline{r}}}{m(r)c^2} \right)^{-1/2} \quad -(13)$$

It is clear that

$$F(\text{spacetime}) \rightarrow \infty \quad -(14)$$

when

$$\frac{dm(r)}{dr} = \frac{2m(r)}{r}, \quad -(15)$$

and in general the maxima and minima are found when:

$$\frac{dF(\text{spacetime})}{dr} = 0 \quad -(16)$$

If d^2F/dr^2 is negative at this point there is a maximum, and if d^2F/dr^2 is positive there is a minimum. If $d^2F/dr^2 = 0$ there can be an inflexion. Therefore it will be interesting to find all the turning points of the function

) given in eq. (12). As in Note 417(5), the lagrangian can be expanded using:

$$-mc^2\left(n(r) - \frac{v^2}{n(r)c^2}\right)^{1/2} \sim \frac{1}{2}\frac{mv^2}{n(r)^{3/2}} - n(r)^{1/2}mc^2 \quad -(17)$$

Similar considerations can be given to the hamiltonian:

$$H = n(r)\gamma mc^2 - n(r)^{1/2}\frac{mMG}{r} \quad -(18)$$

and the angular momentum:

$$L = \frac{\gamma mc^2}{n(r)}\dot{\phi} \quad -(19)$$

Consider the hamiltonian:

$$\begin{aligned} H_0 &= H - mc^2 \\ &= (\gamma n(r) - 1)mc^2 - n(r)^{1/2}\frac{mMG}{r} \\ &= T + U \quad -(20) \end{aligned}$$

where T is the kinetic energy and U is the potential energy, and define:

$$n(r) = 1 - f(r) \quad -(21)$$

The hamiltonian in m space is therefore:

$$H_0 = (\gamma - 1)mc^2 - f(r)mc^2 - \frac{mMG}{r}(1 - f(r))^{1/2} \quad -(22)$$

and if

$$f(r) << 1 \quad -(23)$$

$$\begin{aligned} H_0 &= (\gamma-1)mc^2 - f(r)mc^2 - \frac{mMG}{r}\left(1 - \frac{1}{2}f(r) + \ldots\right) \quad -(24) \\ &= T + U + H_0(\text{spacetime}) \end{aligned}$$

with:

$$\gamma = \left(n(r) - \frac{v^2}{n(r)c^2}\right)^{-1/2} \quad -(25)$$

In these equations:

$$T = (\gamma - 1) mc^2 \quad -(26)$$

the relativistic kinetic energy,

$$U = -\frac{mMG}{r} \quad -(27)$$

the gravitational potential, and

$$\boxed{H_0(\text{spacetime}) = -f(r)\, mc^2 + \frac{1}{2}\frac{mMG}{r} f(r)} \quad -(28)$$

the spacetime energy due to the function $f(r)$. The
spacetime energy does not appear in classical dynamics
d orbit theory, and does not appear in Minkowski spacetime,
which

$$f(r)(\text{Minkowski}) = 0. \quad -(29)$$

Now expand the generalized Lorentz factor:

$$\gamma = \left(n(r) - \frac{v^2}{n(r)c^2}\right)^{-1/2} = n(r)^{-1/2}\left(1 - \left(\frac{v}{n(r)c}\right)^2\right)^{-1/2} \quad -(30)$$

and it follows that the hamiltonian can be expressed as:

$$H = n(r)^{1/2}\left(1 - \left(\frac{v}{n(r)c}\right)^2\right)^{-1/2} mc^2 - n(r)^{1/2}\frac{mMG}{r}$$

$$= n(r)^{1/2}\left[\left(1 - \left(\frac{v}{n(r)c}\right)^2\right)^{-1/2} mc^2 - \frac{mMG}{r}\right]$$

$$\sim n(r)^{1/2}\left[\left(1 + \frac{1}{2}\frac{v^2}{n^2(r)c^2}\right)mc^2 - \frac{mMG}{r}\right] \quad -(31)$$

Now define:

$$H_0 := H - n(r)^{1/2} mc^2 \quad -(32)$$

so:

$$H_0 := H - mc^2 m(r)^{1/2} = \frac{1}{2}\frac{mv^2}{m(r)^{3/2}} - m(r)^{1/2}\frac{mMG}{r} \quad -(33)$$

Similarly, if we define the lagrangian:

$$\mathcal{L}_0 = \mathcal{L} + mc^2 m(r)^{1/2} \quad -(34)$$

it is found that:

$$\boxed{\mathcal{L}_0 = H_0 = \frac{1}{2}\frac{mv^2}{m(r)^{3/2}} + m(r)^{1/2}\frac{mMG}{r}} \quad -(35)$$

This hamiltonian can be developed as an orbit theory:

$$H_0 = \frac{1}{2}\frac{m(\dot{r}^2 + r^2\dot{\phi}^2)}{m(r)^{3/2}} - m(r)^{1/2}\frac{mMG}{r} \quad -(36)$$

$$= \frac{1}{2}\frac{m\dot{r}^2}{m(r)^{3/2}} + \frac{1}{2}\frac{mr^2\dot{\phi}^2}{m(r)^{3/2}} - m(r)^{1/2}\frac{mMG}{r}$$

The angular momentum is defined as:

$$L = \frac{\gamma m r^2 \dot{\phi}}{m(r)} \quad -(37)$$

and it follows that:

$$\dot{\phi}^2 = \frac{m(r)^2 L^2}{\gamma^2 m^2 r^4} \quad -(38)$$

Therefore:

$$H_0 = \frac{1}{2}\frac{m\dot{r}^2}{m(r)^{3/2}} + \frac{1}{2}m(r)^{1/2}\frac{L^2}{\gamma^2 m r^2} - m(r)^{1/2}\frac{mMG}{r} \quad -(39)$$

Now use:

$$\frac{1}{\gamma^2} = m(r) - \frac{v^2}{c^2 m(r)} \xrightarrow[v \ll c]{} m(r) \quad -(40)$$

to obtain:

$$H_0 \sim \frac{1}{2}\frac{m\dot{r}^2}{n(r)^{3/2}} + \frac{1}{2}n(r)^{3/2}\frac{L^2}{mr^2} - n(r)^{1/2}\frac{mMG}{r} \quad -(41)$$

In the limit:

$$n(r) \to 1 \quad -(42)$$

Eq. (41) correctly reduces to the classical limit:

$$H_0 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} - \frac{mMG}{r} \quad -(43)$$

Eq. (43) gives the conic section orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(44)$$

So eq. (41) will lead to an analytically tractable modification of the orbit (44).

This is useful in order to check the numerical results from the solution of:

$$\frac{dH}{dt} = 0 \quad -(45)$$

and

$$\frac{dL}{dt} = 0 \quad -(46)$$

From eq. (41):

$$\left(\frac{dr}{dt}\right)^2 = \frac{2n(r)^{3/2}}{m}\left(H_0 - \frac{1}{2}n(r)^{3/2}\frac{L^2}{mr^2} + n(r)^{1/2}\frac{mMG}{r}\right) \quad -(47)$$

So dr/dt can be computed from eq. (47) in the classical approximation (40), and compared with the classical:

$$\left(\frac{dr}{dt}\right)_0^2 = \frac{2}{m}\left(H_0 - \frac{1}{2}\frac{L^2}{mr^2} + \frac{mMG}{r}\right) - (48)$$

This illustrates the way in which the function $n(r)$ affects well known orbit theory in the classical limit.

The approximation (40) is useful to remove a recursive problem in the calculation, thereby simplifying it greatly.

The orbital velocity is given by:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad -(49)$$

$$= \left(\frac{dr}{dt}\right)^2 + \frac{m(r)^2 L^2}{\gamma^2 m^2 r^2}$$

$$\sim \left(\frac{dr}{dt}\right)^2 + \frac{m(r)^3 L^2}{m^2 r^2}$$

in the approximation (40). If this approximation is not used, the calculation becomes considerably more complicated, because γ contains v^2.

So the orbital velocity v is given by:

$$\boxed{v^2 = \frac{2m(r)^{3/2}}{m}\left(H_0 - \frac{1}{2}m(r)^{3/2}\frac{L^2}{mr^2} + m(r)^{1/2}\frac{mMG}{r}\right) + \frac{m(r)L^2}{m^2r^2}}$$

$$-(50)$$

The Newtonian limit is:

$$v_N^2 = \frac{2}{m}\left(H_0 - \frac{L^2}{2mr^2} + \frac{mMG}{r}\right) + \frac{L^2}{m^2r^2}$$

$$= MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(51)$$

$$= 2\left(\frac{H_0}{m} + \frac{MG}{r}\right)$$

Using:

$$H_0 = \frac{1}{2}mv^2 - \frac{nMG}{r} \quad -(52)$$

$$= \frac{1}{2}nMG\left(\frac{2}{r} - \frac{1}{a}\right) - \frac{nMG}{r}$$

$$= -\frac{nMG}{2a}$$

It follows that:

$$v_N^2 = 2\left(\frac{H_0}{m} + \frac{MG}{r}\right) = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(53)$$

Q.E.D.

This well known Newtonian result is modified to eq. (50), which simplifies to:

$$v^2 = \frac{2m(r)^{3/2}}{m}\left(H_0 + m(r)^{1/2}\frac{nMG}{r}\right)$$

$$= 2\left(m(r)^{3/2}\frac{H_0}{m} + m(r)^2\frac{MG}{r}\right)$$

$$\boxed{v^2 = 2m(r)^{3/2}\left(\frac{H_0}{m} + m(r)^{1/2}\frac{MG}{r}\right)} \quad -(54)$$

From eq. (33):

$$H_0 = \frac{1}{2}\frac{mv^2}{m(r)^{3/2}} - m(r)^{1/2}\frac{nMG}{r} \quad -(55)$$

From eqs. (54) and (55):

$$H_0 = H_0 + m(r)^{1/2}\left(\frac{nMG}{r} - \frac{nMG}{r}\right) \quad -(56)$$

so eqs. (54) and (55) are self consistent, Q.E.D.

) This also proves that the methods and approximations used in this note are self consistent.

For small departures from Newtonian orbit theory:

$$H_0 \sim -\frac{mMG}{2a} \quad -(57)$$

So

$$v^2 = 2n(r)^{3/2}\left(n(r)^{1/2}\frac{MG}{r} - \frac{MG}{2a}\right)$$

i.e.

$$v^2 = MG\left(\frac{2n(r)^2}{r} - \frac{n(r)^{3/2}}{a}\right) \quad -(58)$$

The orbital velocity is

$$\boxed{v^2 = n(r)^{3/2} MG\left(\frac{2n(r)^{1/2}}{r} - \frac{1}{a}\right).} \quad -(59)$$

In the Newtonian limit:

$$n(r) \rightarrow 1 \quad -(60)$$

eq. (59) reduces correctly to the Newtonian:

$$v_N^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(61)$$

Eq. (59) gives a method of measuring $n(r)$ by observing the orbital velocity v at a point r in the orbit. Using computer algebra, eq. (59) can be solved for $n(r)$ in terms of v and r. Eq. (59) gives a quartic in $x = n(r)^{1/2}$:

$$\boxed{2ax^4 - rx^3 = \frac{v^2 ra}{MG}} \quad -(62)$$

which reduces to eq. (61) when $x = 1$, Q.E.D.