417(1): Vacuum Force due to Spherically Symmetric Spacetime

Consider the lagrangian of spherically symmetric spacetime:

$$\mathcal{L} = -mc^2 \left(n(r) - \frac{1}{c^2} \dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1 \right)^{1/2} + \frac{mMG}{r_1} \quad -(1)$$

where

$$r_1 = \frac{r}{n(r)^{1/2}} \quad -(2)$$

is given in UFT416. The Euler Lagrange equation is:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}_1} = \frac{\partial \mathcal{L}}{\partial \underline{r}_1} = \underline{\nabla}_1 \mathcal{L} \quad -(3)$$

where

$$\underline{\nabla}_1 \mathcal{L} := \frac{\partial \mathcal{L}}{\partial r_1} \underline{e}_r \quad -(4)$$

By definition:

$$\underline{p}_1 := \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}_1} \quad -(5)$$

is the linear momentum.

The force equation is therefore:

$$\underline{F}_1 = \frac{d\underline{p}_1}{dt} = \frac{\partial \mathcal{L}}{\partial r_1} \underline{e}_r \quad -(6)$$

where

$$\underline{p}_1 = \gamma m \dot{\underline{r}}_1 = \frac{\gamma m \dot{\underline{r}}}{n(r)^{1/2}} \quad -(7)$$

Here:

$$\gamma := \left(n(r) - \frac{1}{c^2} \dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1 \right)^{-1/2} \quad -(8)$$

As spherical spacetime approaches Minkowski spacetime:

$$m(r) \longrightarrow 1 \quad -(9)$$

Therefore:

$$\frac{\partial \mathcal{L}}{\partial r_1} = -\frac{mc^2}{2} \gamma \frac{dm(r)}{dr_1} - \frac{mMG}{r_1^2} \quad -(10)$$

From eqs. (6) and (10):

$$\boxed{\underline{F}_1 = \frac{d\underline{p}_1}{dt} = \left(-\frac{mc^2}{2} \gamma \frac{dm(r)}{dr_1} - \frac{mMG}{r_1^2} \right) \underline{e}_r} \quad -(11)$$

$$= \underline{F} + \underline{F}(\text{vacuum})$$

where

$$\underline{F} = -\frac{mMG}{r_1^2} \underline{e}_r \quad -(12)$$

is the inverse square law in spherically symmetric spacetime, and

$$\underline{F}(\text{vacuum}) = -\frac{mc^2}{2} \gamma \frac{dm(r)}{dr_1} \underline{e}_r \quad -(13)$$

is the force due to spherically symmetric spacetime itself, the vacuum force.

The total force is therefore the ECE2 force

$$\underline{F} = -\underline{\nabla} \underline{\Phi} + \underline{\Omega}_r \underline{\Phi}. \quad -(14)$$

Here $\underline{\Phi}$ is the gravitational potential.

$$\underline{\Phi} = -\frac{mMG}{r} \quad -(15)$$

and $\underline{\Omega}_r$ is the relevant spin connection for spherically symmetric spacetime:

$$\underline{\Omega}_r = \Omega_r \, \underline{e}_r \quad -(16)$$

Therefore the vacuum force is:

$$\boxed{\underline{F}(\text{vacuum}) = \underline{\Omega}_r \, \underline{\Phi} = -\frac{mc^2}{2} \gamma \frac{dm(r)}{dr_1} \underline{e}_r} \quad -(17)$$

In this equation:

$$\frac{dm(r)}{dr_1} = \frac{dm(r)}{dr} \frac{dr}{dr_1} \quad -(18)$$

From eq. (2):

$$\frac{dr_1}{dr} = \frac{d}{dr}\left(\frac{r}{m(r)}\right) \quad -(19)$$

$$= \left(m(r) - r\frac{dm(r)}{dr}\right) \Big/ m^2(r)$$

So:

$$\boxed{F(\text{vac}) = -\frac{mc^2}{2} \frac{\gamma}{m^2(r)} \frac{dm(r)}{dr} \left(m(r) - \frac{dm(r)}{dr}\right)} \quad -(20)$$

the magnitude of the vacuum force due to spherically symmetric spacetime.

In eq. (20):

$$\gamma = \left(n(r) - \frac{v^2}{n(r)c^2} \right)^{-1/2} \quad -(21)$$

here in (r, ϕ) coordinates:

$$v^2 = \dot{r}^2 + r^2 \dot{\phi}^2 \quad -(22)$$

, the Newtonian velocity.

If:

$$n(r) = 1 \quad -(23)$$

then

$$F(\text{vac}) = 0 \quad -(24)$$

and the orbit is an ellipse or conic section. In general the orbit is given by numerical solution of

$$\frac{dH}{dt} = 0 \quad -(25)$$

and

$$\frac{dL}{dt} = 0 \quad -(26)$$

where:

$$H = \gamma n(r) mc^2 \quad -(27)$$

and

$$L = \frac{\gamma m r^2}{n(r)} \dot{\phi} \quad -(28)$$

These concepts are consistent with the fact that spin connection of ECE2 describes departures from Minkowski spacetime, which is turn define vacuum force and vacuum energy and any non Newtonian orbit.