6(3): Self Consistency with the Geodesic Method

Consider the infinitesimal line element of m theory:

$$ds^2 = m(r)c^2dt^2 - \frac{dr^2}{m(r)} - r^2 d\phi^2 \quad - (1)$$

$$= c^2 d\tau^2$$

In the (r_1, ϕ) coordinate system:

$$r_1 = \frac{r}{m(r)^{1/2}} \quad - (2)$$

and

$$ds^2 = c^2 d\tau^2 = m(r_1)c^2dt^2 - dr_1^2 - r_1^2 d\phi^2 \quad - (3)$$

From eq. (3):

$$mc^2 = m\,m(r_1)c^2\left(\frac{dt}{d\tau}\right)^2 - m\left(\frac{dr_1}{d\tau}\right)^2 - mr_1^2\left(\frac{d\phi}{d\tau}\right)^2 \quad - (4)$$

In Eq. (3):

$$v^2dt^2 = dr_1^2 + r_1^2 d\phi^2 \quad - (5)$$

so

$$c^2d\tau^2 = m(r_1)c^2dt^2 - v^2dt^2 \quad - (6)$$

$$= (m(r_1)c^2 - v^2)dt^2$$

Therefore the Lorentz factor is generalized to:

$$\gamma = \frac{dt}{d\tau} = \left(\frac{c^2}{m(r_1)c^2 - v^2}\right)^{1/2}$$

$$= \left(m(r_1) - \frac{v^2}{c^2}\right)^{-1/2} \quad - (7)$$

so eq. (4) becomes:

$$mc^2 = mc^2 m(r_1)\gamma^2 - \gamma^2 m\left(\dot{r}_1^2 + r_1^2\dot{\phi}^2\right) \quad - (8)$$

so

$$m(r_1)m^2c^4 = m^2c^4 m(r_1)^2\gamma^2 - \gamma^2 c^2 m^2 m(r)\left(\dot{r}_1^2 + r_1^2\dot{\phi}^2\right) \quad - (9)$$

$$= E^2 - c^2p^2$$

) where the total relativistic energy is:

$$E = m(r_1)\gamma m c^2 \quad -(10)$$

and where the relativistic momentum is defined by:

$$p^2 = m^2\gamma^2\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \quad -(11)$$

It follows that:

$$\boxed{m^2c^4 = \frac{1}{m(r)}\left(E^2 - c^2p^2\right)} \quad -(12)$$

This is the Einstein energy equation in m space.

Eq. (12) can be considered to be a geodesic equation

$$\epsilon = g^{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} := \frac{E^2 - c^2p^2}{m(r)} = m^2c^4 \quad -(13)$$

The infinitesimal line element (3) can be written as:

$$c^2 = m(r_1)c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr_1}{d\tau}\right)^2 - r_1^2\left(\frac{d\phi}{d\tau}\right)^2 \quad -(14)$$

and as given in Carroll chapter 7 this can be considered to be a geodesic equation:

$$c^2 = \epsilon = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \text{constant of motion} \quad -(15)$$

$$= m(r)\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{1}{m(r)}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right)$$

In analogy with:

$$T = \frac{1}{2}mv^2 \quad -(16)$$

a free particle kinetic lagrangian can be defined: $-(16)$

$$\mathcal{L} = \frac{1}{2}mc^2 = \frac{1}{2}m\left(\frac{ds}{d\tau}\right)^2 = \frac{1}{2}m g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\mu}{d\tau}$$

) Using eq. (14):

$$\mathcal{L} = \frac{1}{2} m \left(n(r_1) c^2 \left(\frac{dt}{d\tau} \right)^2 - \left(\frac{dr_1}{d\tau} \right)^2 - r_1^2 \left(\frac{d\phi}{d\tau} \right)^2 \right)$$

$$= \frac{1}{2} m g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \quad -(17)$$

$$= \frac{1}{2} m \left(g_{00} \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} - g_{11} \frac{dx^1}{d\tau} \frac{dx^1}{d\tau} - g_{22} \frac{dx^2}{d\tau} \frac{dx^2}{d\tau} \right)$$

Therefore:

$$\frac{1}{2} m g_{00} \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} = \frac{1}{2} m n(r) c^2 \left(\frac{dt}{d\tau} \right)^2 \quad -(18)$$

$$\frac{1}{2} m g_{11} \frac{dx^1}{d\tau} \frac{dx^1}{d\tau} = \frac{1}{2} m \left(\frac{dr_1}{d\tau} \right)^2 \quad -(19)$$

$$\frac{1}{2} m g_{22} \frac{dx^2}{d\tau} \frac{dx^2}{d\tau} = \frac{1}{2} m \left(\frac{d\phi}{d\tau} \right)^2 r_1^2 \quad -(20)$$

The Hamilton Principle of Least Action is:

$$\delta \int \mathcal{L} \, d\tau = 0 \quad -(21)$$

and the Euler Lagrange equation is:

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} \quad -(22)$$

where:

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} \quad -(23)$$

It follows that:

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^0} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial(dt/d\tau)} = 0 \quad -(24)$$

i.e.

$$\frac{dE}{d\tau} = 0 \quad -(25)$$

where

$$E = \frac{\partial \mathcal{L}}{\partial(dt/d\tau)} = n(r_1)mc^2\frac{dt}{d\tau} \quad -(26)$$

$$= n(r_1)\gamma mc^2$$

is the total relativistic energy of a free particle. So

$$\boxed{E = n(r_1)\gamma mc^2} \quad -(27)$$

is a constant of motion for a free particle. In general, the hamiltonian is:

$$H = E + U = T + U \quad -(28)$$

$$\boxed{H = n(r_1)\gamma mc^2 - \frac{mMG}{r_1}} \quad -(29)$$

The hamiltonian is a constant of motion in general:

$$\frac{dH}{dt} = 0 \quad -(30)$$

as in UFT 415, Q.E.D.

Also:

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^1} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial(dr_1/d\tau)} = 0 \quad -(31)$$

i.e.

$$\frac{dp_1}{d\tau} = 0 \quad -(32)$$

here

$$p_1 = \frac{\partial \mathcal{L}}{\partial(dr_1/d\tau)} = m\frac{dr_1}{d\tau} \quad -(33)$$

is the linear momentum of a free particle.

By definition:

$$p = m\frac{dr_1}{d\tau} = \frac{\gamma}{m(r_1)^{1/2}} m\frac{dr}{dt} \quad -(34)$$

s in 4FT415, Q.E.D.

Finally:

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^2} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial(d\phi/d\tau)} = 0 \quad -(35)$$

i.e

$$\frac{dL}{d\tau} = 0 \quad -(36)$$

where

$$L = m r_1^2 \frac{d\phi}{d\tau} \quad -(37)$$

is the angular momentum of a free particle.

By definition:

$$\boxed{L = m r_1^2 \frac{d\phi}{d\tau} = \frac{\gamma m r^2}{m(r)} \frac{d\phi}{dt}} \quad -(38)$$

g in 4FT415, Q.E.D.

The equations of motion of the new cosmology are therefore:

$$\boxed{\begin{array}{l} \dfrac{dH}{d\tau} = 0 \quad -(39) \\ \dfrac{dL}{d\tau} = 0 \quad -(40) \end{array}}$$

Finally we:

$$\frac{dH}{d\tau} = \frac{dH}{dt}\frac{dt}{d\tau} = \gamma \frac{dH}{dt} \quad -(41)$$

nd

$$\frac{dL}{d\tau} = \frac{dL}{dt}\frac{dt}{d\tau} = \gamma \frac{dL}{dt} \quad -(42)$$

nd it follows that:

$$\boxed{\begin{aligned} \frac{dH}{dt} &= 0 \quad -(43) \\ \frac{dL}{dt} &= 0 \quad -(44) \end{aligned}}$$

s is UFT415, Q.E.D.