1 Coordinates

```
(%i1)  kill(all);
(%o0)  done
```

```
(%i1)  depends([phi,r,omega,T,U,L,gamma],t,[Phi],r);
(%o1)  [φ(t),r(t),ω(t),T(t),U(t),L(t),Γ(t),Φ(r)]
```

```
(%i2)  assume(c>0, Gamma>0);
(%o2)  [c>0,Γ>0]
```

Relativistic Lagrangian

1 Equivalent of kinetic energy

```
(%i3)  T: -m*c^2/gamma;
```

$$(T)\quad -\frac{c^2\,m}{\Gamma}$$

```
(%i4)  v2: diff(r,t)^2+r^2*diff(phi,t)^2;
```

$$(v2)\quad \left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,r\right)^2+\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,\varphi\right)^2 r^2$$

```
(%i5)  gamma: 1/sqrt(1-v2/c^2);
```

$$(gamma)\quad \frac{1}{\sqrt{1-\frac{\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,r\right)^2+\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,\varphi\right)^2 r^2}{c^2}}}$$

2 Potential energy

```
(%i6)  Phi: -M*G/r;
```

$$(Phi)\quad -\frac{G\,M}{r}$$

```
(%i7)  Ustd: m*Phi;
```

$$(Ustd)\quad -\frac{G\,M\,m}{r}$$

3 Lagrangian

```
(%i8)  La: ev(T)-Ustd;
```

$$(La)\quad \frac{G\,M\,m}{r}-c^2\,m\sqrt{1-\frac{\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,r\right)^2+\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,\varphi\right)^2 r^2}{c^2}}$$

```
(%i9)  Etot: T+Ustd;
```

$$(Etot)\quad -\frac{c^2\,m}{\Gamma}-\frac{G\,M\,m}{r}$$

4 Lagrange equations

4.1 Definitions

```
(%i10)  D1(La,q) := diff(La, q)/*+Q[q]*/;
```

$$(\%o10)\quad \mathrm{D1}(La,q):=\frac{\mathrm{d}}{\mathrm{d}\,q}\,La$$

```
(%i11)  D2a(La,q) := diff(La, diff(q,t));/*only for constants of motion*/;
```

$$(\%o11)\quad \mathrm{D2a}(La,q):=\frac{\mathrm{d}}{\mathrm{d}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,q\right)}\,La$$

```
(%i12)  D2(La,q) := diff(diff(La, diff(q,t)),t);
```

$$(\%o12)\quad \mathrm{D2}(La,q):=\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\mathrm{d}}{\mathrm{d}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,q\right)}\,La\right)$$

```
(%i13)  LE(La,q) := D1(La,q) = D2(La,q);
```

(%o13) $\mathrm{LE}(La,q):=\mathrm{D1}(La,q)=\mathrm{D2}(La,q)$

4.2 Evaluation

```
(%i14)  E11: LE(La,phi);
```

(E11) $$0=\frac{m\left(\frac{d}{dt}\varphi\right)r^2\left(2\left(\frac{d}{dt}r\right)\left(\frac{d^2}{dt^2}r\right)+2\left(\frac{d}{dt}\varphi\right)^2 r\left(\frac{d}{dt}r\right)+2\left(\frac{d}{dt}\varphi\right)\left(\frac{d^2}{dt^2}\varphi\right)r^2\right)}{2c^2\left(1-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2}\right)^{3/2}}+\frac{2m\left(\frac{d}{dt}\varphi\right)r\left(\frac{d}{dt}r\right)}{\sqrt{1-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2}}}+\frac{m\left(\frac{d^2}{dt^2}\varphi\right)r^2}{\sqrt{1-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2}}}$$

```
(%i15)  E21: LE(La,r);
```

(E21) $$\frac{m\left(\frac{d}{dt}\varphi\right)^2 r}{\sqrt{1-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2}}}-\frac{GMm}{r^2}=\frac{m\left(\frac{d}{dt}r\right)\left(2\left(\frac{d}{dt}r\right)\left(\frac{d^2}{dt^2}r\right)+2\left(\frac{d}{dt}\varphi\right)^2 r\left(\frac{d}{dt}r\right)+2\left(\frac{d}{dt}\varphi\right)\left(\frac{d^2}{dt^2}\varphi\right)r^2\right)}{2c^2\left(1-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2}\right)^{3/2}}+\frac{m\left(\frac{d^2}{dt^2}r\right)}{\sqrt{1-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2}}}$$

4.3 Constants of motion

```
(%i16)  D2a(La, phi);
```

(%o16) $$\frac{m\left(\frac{d}{dt}\varphi\right)r^2}{\sqrt{1-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2}}}$$

4.4 Resolving for highest derivatives of variables and final equations

```
(%i17)  GC: trigsimp(first(solve([E11,E21],
            [diff(phi,t,2), diff(r,t,2)])));
```

(GC) $$\left[\frac{d^2}{dt^2}\varphi=\frac{GM\left(\frac{d}{dt}\varphi\right)\left(\frac{d}{dt}r\right)\sqrt{-\left(\frac{d}{dt}r\right)^2-\left(\frac{d}{dt}\varphi\right)^2 r^2+c^2}-2c^3\left(\frac{d}{dt}\varphi\right)r\left(\frac{d}{dt}r\right)}{c^3 r^2},\frac{d^2}{dt^2}r=\frac{\sqrt{-\left(\frac{d}{dt}r\right)^2-\left(\frac{d}{dt}\varphi\right)^2 r^2+c^2}\left(GM\left(\frac{d}{dt}r\right)^2-GMc^2\right)+c^3\left(\frac{d}{dt}\varphi\right)^2 r^3}{c^3 r^2}\right]$$

4.5 Re-inserting gamma

```
(%i18)  expand(ratsubst(c/%gamma, sqrt(-('diff(r,t,1))^2-('diff(phi,t,1))^2*r^2+c^2),GC));
```

(%o18) $$\left[\frac{d^2}{dt^2}\varphi=\frac{GM\left(\frac{d}{dt}\varphi\right)\left(\frac{d}{dt}r\right)}{\gamma c^2 r^2}-\frac{2\left(\frac{d}{dt}\varphi\right)\left(\frac{d}{dt}r\right)}{r},\frac{d^2}{dt^2}r=\frac{GM\left(\frac{d}{dt}r\right)^2}{\gamma c^2 r^2}+\left(\frac{d}{dt}\varphi\right)^2 r-\frac{GM}{\gamma r^2}\right]$$

```
(%i19)  factor(%);
```

(%o19) $$\left[\frac{d^2}{dt^2}\varphi=-\frac{\left(\frac{d}{dt}\varphi\right)(2\gamma c^2 r-GM)\left(\frac{d}{dt}r\right)}{\gamma c^2 r^2},\frac{d^2}{dt^2}r=\frac{GM\left(\frac{d}{dt}r\right)^2+\gamma c^2\left(\frac{d}{dt}\varphi\right)^2 r^3-GMc^2}{\gamma c^2 r^2}\right]$$

Relativistic Lagrangian with m(r)

```
(%i20)  kill(gamma,v2);
(%o20)  done
```

1 Equivalent of kinetic energy

```
(%i21)  T: -/*sqrt(m(r))**/m*c^2/gamma;
```

$$(T) \quad -\frac{c^2\, m}{gamma}$$

```
(%i22)  v2: (diff(r,t)^2+r^2*diff(phi,t)^2)/m(r);
```

$$(v2) \quad \frac{\left(\frac{d}{d\,t}\,r\right)^2+\left(\frac{d}{d\,t}\,\varphi\right)^2 r^2}{m(r)}$$

```
(%i23)  gamma: 1/sqrt(m(r)-v2/c^2);
```

$$(gamma) \quad \frac{1}{\sqrt{m(r)-\frac{\left(\frac{d}{d\,t}\,r\right)^2+\left(\frac{d}{d\,t}\,\varphi\right)^2 r^2}{c^2\, m(r)}}}$$

2 Potential energy

```
(%i24)  Phi: -sqrt(m(r))*M*G/r;
```

$$(Phi) \quad -\frac{G\,M\sqrt{m(r)}}{r}$$

```
(%i25)  Ustd: m*Phi;
```

$$(Ustd) \quad -\frac{G\,M\,m\sqrt{m(r)}}{r}$$

3 Lagrangian

```
(%i26)  La: ev(T)-Ustd;
```

$$(La) \quad \frac{G\,M\,m\sqrt{m(r)}}{r}-c^2\, m\sqrt{m(r)-\frac{\left(\frac{d}{d\,t}\,r\right)^2+\left(\frac{d}{d\,t}\,\varphi\right)^2 r^2}{c^2\, m(r)}}$$

```
(%i27)  Etot: T+Ustd;
```

$$(Etot) \quad -\frac{c^2\, m}{gamma}-\frac{G\,M\,m\sqrt{m(r)}}{r}$$

3.1 Evaluation

```
(%i28)  E11: LE(La,phi);
```

$$(E11) \quad 0=-\frac{m\left(\frac{d}{d\,t}\,\varphi\right) r^2\left(\frac{\left(\left(\frac{d}{d\,t}\,r\right)^2+\left(\frac{d}{d\,t}\,\varphi\right)^2 r^2\right)\left(\frac{d}{d\,t}\,m(r)\right)}{c^2\, m(r)^2}+\frac{d}{d\,t}\,m(r)-\frac{2\left(\frac{d}{d\,t}\,r\right)\left(\frac{d^2}{d\,t^2}\,r\right)+2\left(\frac{d}{d\,t}\,\varphi\right)^2 r\left(\frac{d}{d\,t}\,r\right)+2\left(\frac{d}{d\,t}\,\varphi\right)\left(\frac{d^2}{d\,t^2}\,\varphi\right) r^2}{c^2\, m(r)}\right)}{2\, m(r)\left(m(r)-\frac{\left(\frac{d}{d\,t}\,r\right)^2+\left(\frac{d}{d\,t}\,\varphi\right)^2 r^2}{c^2\, m(r)}\right)^{3/2}}-\frac{m\left(\frac{d}{d\,t}\,\varphi\right) r^2\left(\frac{d}{d\,t}\,m(r)\right)}{m(r)^2\sqrt{m(r)-\frac{\left(\frac{d}{d\,t}\,r\right)^2+\left(\frac{d}{d\,t}\,\varphi\right)^2 r^2}{c^2\, m(r)}}}+\frac{2\, m\left(\frac{d}{d\,t}\,\varphi\right) r\left(\frac{d}{d\,t}\,r\right)}{m(r)\sqrt{m(r)-\frac{\left(\frac{d}{d\,t}\,r\right)^2+\left(\frac{d}{d\,t}\,\varphi\right)^2 r^2}{c^2\, m(r)}}}+\frac{m\left(\frac{d^2}{d\,t^2}\,\varphi\right) r^2}{m(r)\sqrt{m(r)-\frac{\left(\frac{d}{d\,t}\,r\right)^2+\left(\frac{d}{d\,t}\,\varphi\right)^2 r^2}{c^2\, m(r)}}}$$

(%i29) `E21: LE(La,r);`

(E21)
$$-\frac{c^2\, m\left(\frac{\left(\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2\right)\left(\frac{d}{dr}m(r)\right)}{c^2\, m(r)^2}+\frac{d}{dr}m(r)-\frac{2\left(\frac{d}{dt}\varphi\right)^2 r}{c^2\, m(r)}\right)}{2\sqrt{m(r)-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2\, m(r)}}}+\frac{G\,M\,m\left(\frac{d}{dr}m(r)\right)}{2\,r\sqrt{m(r)}}-\frac{G\,M\,m\sqrt{m(r)}}{r^2}=-$$
$$\frac{m\left(\frac{d}{dt}r\right)\left(\frac{\left(\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2\right)\left(\frac{d}{dt}m(r)\right)}{c^2\, m(r)^2}+\frac{d}{dt}m(r)-\frac{2\left(\frac{d}{dt}r\right)\left(\frac{d^2}{dt^2}r\right)+2\left(\frac{d}{dt}\varphi\right)^2 r\left(\frac{d}{dt}r\right)+2\left(\frac{d}{dt}\varphi\right)\left(\frac{d^2}{dt^2}\varphi\right)r^2}{c^2\, m(r)}\right)}{2\,m(r)\left(m(r)-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2\, m(r)}\right)^{3/2}}-$$
$$\frac{m\left(\frac{d}{dt}r\right)\left(\frac{d}{dt}m(r)\right)}{m(r)^2\sqrt{m(r)-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2\, m(r)}}}+\frac{m\left(\frac{d^2}{dt^2}r\right)}{m(r)\sqrt{m(r)-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2\, m(r)}}}$$

3.2 Removing dm(r)/dt

➔
```
/*E11: ratsubst(0, diff(m(r),t), E11);
E21: ratsubst(0, diff(m(r),t), E21);*/;
```

3.3 Replacing dm(r)/dt

```
(%i31)  E11: ratsubst(diff(r,t)*diff(m(r),r), diff(m(r),t), E11);
        E21: ratsubst(diff(r,t)*diff(m(r),r), diff(m(r),t), E21);
```

(E11)
$$0=-\left(c^5 m\left(\frac{d}{dt}\varphi\right) r^2\left(\frac{d}{dt}r\right)^3-2c^5 m\left(\frac{d}{dt}\varphi\right) r^2\left(\frac{d}{dt}r\right)\left(\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2\right)+\left(c^7 m\left(\frac{d}{dt}\varphi\right) r^2\,m(r)^2+c^5 m\left(\frac{d}{dt}\varphi\right)^3 r^4\right)\left(\frac{d}{dt}r\right)\right)\left(\frac{d}{dr}m(r)\right)-2c^5 m\left(\frac{d}{dt}\varphi\right) r^2\,m(r)\left(\frac{d}{dt}r\right)\left(\frac{d^2}{dt^2}r\right)-\frac{\left(-4c^5 m\left(\frac{d}{dt}\varphi\right) r\,m(r)^2\left(\frac{d}{dt}r\right)-2c^5 m\left(\frac{d^2}{dt^2}\varphi\right) r^2\,m(r)^2\right)\left(\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2\right)}{m(r)}-2c^5 m\left(\frac{d}{dt}\varphi\right)^3 r^3\,m(r)\left(\frac{d}{dt}r\right)-2c^5 m\left(\frac{d}{dt}\varphi\right)^2\left(\frac{d^2}{dt^2}\varphi\right) r^4\,m(r))/(2c^4\,m(r)^3\left(-\frac{\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{m(r)}\right)^{3/2})$$

(E21)
$$\left(GMcmr\,m(r)^2\sqrt{-\frac{\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{m(r)}}-c^2 m r^2\sqrt{m(r)}\left(\frac{d}{dt}r\right)^2-c^4 m r^2\,m(r)^{5/2}-c^2 m\left(\frac{d}{dt}\varphi\right)^2 r^4\sqrt{m(r)}\right)\left(\frac{d}{dr}m(r)\right)-2GMcm\,m(r)^3\sqrt{-\frac{\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{m(r)}}+2c^2 m\left(\frac{d}{dt}\varphi\right)^2 r^3\,m(r)^{3/2})/(2c\,r^2\,m(r)^{5/2}\sqrt{-\frac{\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{m(r)}})=-\left(c^3 m\left(\frac{d}{dt}r\right)^4-2c^3 m\left(\frac{d}{dt}r\right)^2\left(\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2\right)+\left(c^5 m\,m(r)^2+c^3 m\left(\frac{d}{dt}\varphi\right)^2 r^2\right)\left(\frac{d}{dt}r\right)^2\right)\left(\frac{d}{dr}m(r)\right)+\left(2c^3 m\,m(r)\left(\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2\right)-2c^3 m\,m(r)\left(\frac{d}{dt}r\right)^2\right)\left(\frac{d^2}{dt^2}r\right)-2c^3 m\left(\frac{d}{dt}\varphi\right)^2 r\,m(r)\left(\frac{d}{dt}r\right)^2-2c^3 m\left(\frac{d}{dt}\varphi\right)\left(\frac{d^2}{dt^2}\varphi\right) r^2\,m(r)\left(\frac{d}{dt}r\right))/(2c^2\,m(r)^3\left(-\frac{\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{m(r)}\right)^{3/2})$$

3.4 Constants of motion

```
(%i32)  D2a(La, phi);
```

(%o32)
$$\frac{m\left(\frac{d}{dt}\varphi\right) r^2}{m(r)\sqrt{m(r)-\frac{\left(\frac{d}{dt}r\right)^2+\left(\frac{d}{dt}\varphi\right)^2 r^2}{c^2\,m(r)}}}$$

3.5 Resolving for highest derivatives of variables and final equations

```
(%i33)  GC: trigsimp(first(solve([E11/gamma^2,E21/gamma^2],
            [diff(phi,t,2), diff(r,t,2)])));
```

(GC)

$$\left[\frac{d^2}{dt^2}\varphi=\left(\left(GM\left(\frac{d}{dt}\varphi\right)r\sqrt{m(r)}\left(\frac{d}{dt}r\right)\left(-\frac{\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2r^2}{m(r)}\right)^{3/2}+4\,c^3\left(\frac{d}{dt}\varphi\right)r^2\left(\frac{d}{dt}r\right)^3+\left(4\,c^3\left(\frac{d}{dt}\varphi\right)^3r^4-4\,c^5\left(\frac{d}{dt}\varphi\right)r^2\,m(r)^2\right)\left(\frac{d}{dt}r\right)\right)\left(\frac{d}{dr}m(r)\right)-2\,GM\left(\frac{d}{dt}\varphi\right)m(r)^{3/2}\left(\frac{d}{dt}r\right)\left(-\frac{\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2r^2}{m(r)}\right)^{3/2}-4\,c^3\left(\frac{d}{dt}\varphi\right)r\,m(r)\left(\frac{d}{dt}r\right)^3+\left(4\,c^5\left(\frac{d}{dt}\varphi\right)r\,m(r)^3-4\,c^3\left(\frac{d}{dt}\varphi\right)^3r^3\,m(r)\right)\left(\frac{d}{dt}r\right)\right)/\left(2\,c^3r^2\,m(r)\left(\frac{d}{dt}r\right)^2-2\,c^5r^2\,m(r)^3+2\,c^3\left(\frac{d}{dt}\varphi\right)^2r^4\,m(r)\right),\ \frac{d^2}{dt^2}r=\left(\left(3\,c^3r^2\left(\frac{d}{dt}r\right)^4+\left(-\frac{\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2r^2}{m(r)}\right)^{3/2}\left(GMr\sqrt{m(r)}\left(\frac{d}{dt}r\right)^2-GMc^2r\,m(r)^{5/2}\right)+\left(2\,c^3\left(\frac{d}{dt}\varphi\right)^2r^4-4\,c^5r^2\,m(r)^2\right)\left(\frac{d}{dt}r\right)^2+c^7r^2\,m(r)^4-c^3\left(\frac{d}{dt}\varphi\right)^4r^6\right)\left(\frac{d}{dr}m(r)\right)+\left(-\frac{\left(\frac{d}{dt}r\right)^2-c^2\,m(r)^2+\left(\frac{d}{dt}\varphi\right)^2r^2}{m(r)}\right)^{3/2}\left(2\,GMc^2\,m(r)^{7/2}-2\,GM\,m(r)^{3/2}\left(\frac{d}{dt}r\right)^2\right)+2\,c^3\left(\frac{d}{dt}\varphi\right)^2r^3\,m(r)\left(\frac{d}{dt}r\right)^2-2\,c^5\left(\frac{d}{dt}\varphi\right)^2r^3\,m(r)^3+2\,c^3\left(\frac{d}{dt}\varphi\right)^4r^5\,m(r)\right)/\left(2\,c^3r^2\,m(r)\left(\frac{d}{dt}r\right)^2-2\,c^5r^2\,m(r)^3+2\,c^3\left(\frac{d}{dt}\varphi\right)^2r^4\,m(r)\right)\right]$$

3.6 Re-inserting gamma

```
(%i34)  GC1: ratsubst(c^2/Gamma^2, c^2/gamma^2,GC);
```

(GC1)

$$\left[\frac{d^2}{dt^2}\varphi=\frac{\left(4\,\Gamma\,c^5\left(\frac{d}{dt}\varphi\right)r^2\,m(r)-GMc^3\left(\frac{d}{dt}\varphi\right)r\sqrt{m(r)}\right)\left(\frac{d}{dt}r\right)\left(\frac{d}{dr}m(r)\right)+\left(2\,GMc^3\left(\frac{d}{dt}\varphi\right)m(r)^{3/2}-4\,\Gamma\,c^5\left(\frac{d}{dt}\varphi\right)r\,m(r)^2\right)\left(\frac{d}{dt}r\right)}{2\,\Gamma\,c^5r^2\,m(r)^2},\ \frac{d^2}{dt^2}r=\left(\left(2\,\Gamma^3c^7r^2\,m(r)^3+G\Gamma^2Mc^5r\,m(r)^{5/2}+\left(-G\Gamma^2Mc^5r\sqrt{m(r)}-3\,\Gamma\,c^7r^2\right)m(r)^2+\left(GMc^5r\sqrt{m(r)}-4\,\Gamma^3c^5\left(\frac{d}{dt}\varphi\right)^2r^4\right)m(r)+G\Gamma^2Mc^3\left(\frac{d}{dt}\varphi\right)^2r^3\sqrt{m(r)}\right)\left(\frac{d}{dr}m(r)\right)-2\,G\Gamma^2Mc^5\,m(r)^{7/2}-2\,GMc^5\,m(r)^{5/2}+m(r)^2\left(2\,G\Gamma^2Mc^5\,m(r)^{3/2}+2\,\Gamma^3c^5\left(\frac{d}{dt}\varphi\right)^2r^3\right)-2\,G\Gamma^2Mc^3\left(\frac{d}{dt}\varphi\right)^2r^2\,m(r)^{3/2}\right)/\left(2\,\Gamma^3c^5r^2\,m(r)^2\right)\right]$$

```
(%i35)  GC2: expand(GC1);
```

(GC2)

$$\left[\frac{d^2}{dt^2}\varphi=\frac{2\left(\frac{d}{dt}\varphi\right)\left(\frac{d}{dt}r\right)\left(\frac{d}{dr}m(r)\right)}{m(r)}-\frac{GM\left(\frac{d}{dt}\varphi\right)\left(\frac{d}{dt}r\right)\left(\frac{d}{dr}m(r)\right)}{2\,\Gamma\,c^2r\,m(r)^{3/2}}+\frac{GM\left(\frac{d}{dt}\varphi\right)\left(\frac{d}{dt}r\right)}{\Gamma\,c^2r^2\sqrt{m(r)}}-\frac{2\left(\frac{d}{dt}\varphi\right)\left(\frac{d}{dt}r\right)}{r},\ \frac{d^2}{dt^2}r=c^2\,m(r)\left(\frac{d}{dr}m(r)\right)+\frac{GM\left(\frac{d}{dr}m(r)\right)}{2\,\Gamma^3r\sqrt{m(r)}}-\frac{2\left(\frac{d}{dt}\varphi\right)^2r^2\left(\frac{d}{dr}m(r)\right)}{m(r)}+\frac{GM\left(\frac{d}{dt}\varphi\right)^2r\left(\frac{d}{dr}m(r)\right)}{2\,\Gamma\,c^2\,m(r)^{3/2}}-\frac{3\,c^2\left(\frac{d}{dr}m(r)\right)}{2\,\Gamma^2}-\frac{GM\sqrt{m(r)}}{\Gamma^3r^2}-\frac{GM\left(\frac{d}{dt}\varphi\right)^2}{\Gamma\,c^2\sqrt{m(r)}}+\left(\frac{d}{dt}\varphi\right)^2r\right]$$

4 Lagrange equation solver (Eqs. in Hamilton form)

```
(%i36)  transform(Eq) := (A: ratsubst(phi_d, diff(phi,t), Eq),
            A: ratsubst(phi_dd, diff(phi,t,2), A),
            A: ratsubst(r_d, diff(r,t), A),
            A: ratsubst(r_dd, diff(r,t,2), A)
        )$
```

```
(%i39)  m1(r):=1;
        m1(r):=2-exp(2*exp(-r/R));
        m1(r):=1-2*M*G/(c^2*r)-alpha/r^2;
```

(%o37) $m1(r):=1$

(%o38) $m1(r):=2-\exp\left(2\exp\left(\frac{-r}{R}\right)\right)$

(%o39) $m1(r):=1-\frac{2MG}{c^2 r}+\frac{-\alpha}{r^2}$

```
(%i42)  G1: transform(ev(GC2[1]))$
        G2: transform(ev(GC2[2]))$
        gamma1: transform(ev(gamma));
```

(gamma1) $$\frac{c}{\sqrt{-\frac{r_d^2-c^2\,m(r)^2+\varphi_d^2\,r^2}{m(r)}}}$$

5 Solution

```
➔       /*om[1]: 0.00+0.0075/(3.5*t+1.);
        om[1]: -0.035*exp(-.025*t);
        om[1]: -0.35/(0.2*t+1);
        om[1]: 0.055*t^2;
        om[1]: -0.035*t;
        om[1]: -0.25*exp(-.015*t);
        om[1]: 0;
        dom[1]: diff(om[1],t);*/;
```

```
(%i43)  str: [m=1, G=1, M=10, c=30, omega[1]=om[1], R=0.15, alpha=0.001 ];
```

(str) $[m=1,\ G=1,\ M=10,\ c=30,\ \omega_1=om_1,\ R=0.15,\ \alpha=0.001]$

```
(%i44)  m(r):=ev(m1(r),str);
```

(%o44) $m(r):=\mathrm{ev}(m1(r),\ str)$

```
(%i46)  m(0.01), numer;
        m(1), numer;
```

(%o45) -11.22222222222222

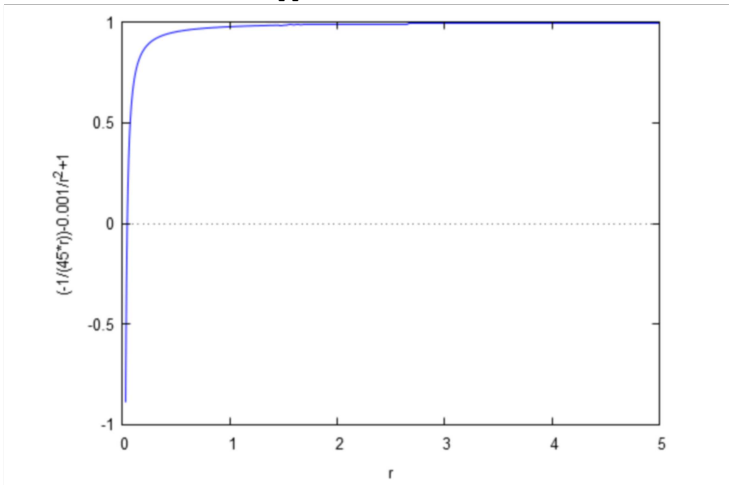
(%o46) 0.9767777777777777

```
(%i47)  Gamma: ev(gamma1,str);
```

(Gamma) $$\frac{30}{\sqrt{-\frac{r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2}{-\frac{1}{45\,r}-\frac{0.001}{r^2}+1}}}$$

```
(%i48)  wxplot2d([m(r)], [r,0,5], [y,-1,1])$
 plot2d: expression evaluates to non-numeric
 value somewhere in plotting range.
plot2d: some values were clipped.
```

(%t48)



5.1 Equation solver

```
(%i52)  Eq1: (ev(rhs(G1), str, diff));
        Eq2: (ev(rhs(G2), str, diff));
        Eq3: phi_d /*phi_dot;*/;
        Eq4: r_d /*r_dot;*/;
```

(Eq1)
$$-\left(r_d\sqrt{-\frac{r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2}{-\frac{1}{45\,r}-\frac{0.001}{r^2}+1}}\left(\left(\frac{1}{45\,r^2}+\frac{0.002}{r^3}\right)\left(10\,\varphi_d\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)r-\frac{108000\,\varphi_d\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^{3/2}r^2}{\sqrt{-\frac{r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2}{-\frac{1}{45\,r}-\frac{0.001}{r^2}+1}}}\right)+\frac{108000\,\varphi_d\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^{5/2}r}{\sqrt{-\frac{r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2}{-\frac{1}{45\,r}-\frac{0.001}{r^2}+1}}}-20\,\varphi_d\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2\right)\right)/\left(54000\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^{5/2}r^2\right)$$

(Eq2)
$$\left(\left(-\frac{r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2}{-\frac{1}{45\,r}-\frac{0.001}{r^2}+1}\right)^{3/2}\left(\left(\frac{1}{45\,r^2}+\frac{0.002}{r^3}\right)\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)\left(-\frac{72900000\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^{3/2}r^2}{\sqrt{-\frac{r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2}{-\frac{1}{45\,r}-\frac{0.001}{r^2}+1}}}-9000\,\varphi_d^2\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)r^3\left(r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2\right)^{-1}\right)-\frac{97200000\,\varphi_d^2\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^{3/2}r^4}{\left(-\frac{r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2}{-\frac{1}{45\,r}-\frac{0.001}{r^2}+1}\right)^{3/2}}+\frac{43740000000\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^{7/2}r^2}{\left(-\frac{r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2}{-\frac{1}{45\,r}-\frac{0.001}{r^2}+1}\right)^{3/2}}+9000\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2r\right)+\frac{48600000\,\varphi_d^2\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^{5/2}r^3}{\left(-\frac{r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2}{-\frac{1}{45\,r}-\frac{0.001}{r^2}+1}\right)^{3/2}}+18000\,\varphi_d^2\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^3r^2\left(r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2\right)^{-1}-18000\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^3\right)\right)/\left(48600000\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^{5/2}r^2\right)$$

(Eq3) φ_d

(Eq4) r_d

Initial values and torques

```
(%i56)  phi_0: 0;
        r_0: 1;
        phi_d0: 3.9;/*3.9*/
        r_d0: 0;
```

(phi_0) 0

(r_0) 1

(phi_d0) 3.9

(r_d0) 0

```
(%i57)  s: rk([Eq1, Eq2, Eq3, Eq4],
            [phi_d, r_d, phi, r],
            [phi_d0, r_d0, phi_0, r_0],
           [t,0,5,0.015])$
```

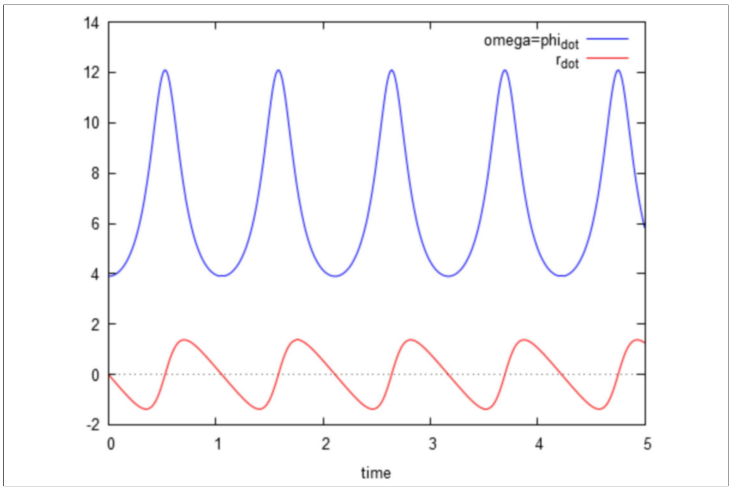
5.2 Graphics

```
(%i61)  c1: makelist([p[1],p[2]],p,s)$
        c2: makelist([p[1],p[3]],p,s)$
        c3: makelist([p[1],p[4]],p,s)$
        c4: makelist([p[1],p[5]],p,s)$
```

5.3 Plot phi, r

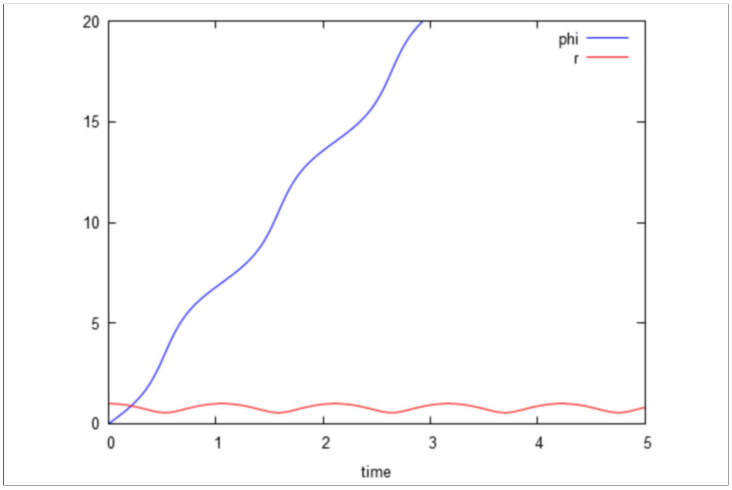
```
(%i62)  wxplot2d([[discrete, c1], [discrete, c2]], /*[y,-10,12],*/
        [xlabel, "time"],
        [legend, "omega=phi_{dot}", "r_{dot}"])$
```

(%t62)




```
(%i63)  wxplot2d([[discrete, c3], [discrete, c4]], [y,0, 20],
        [xlabel, "time"],
        [legend, "phi", "r"])$
```

(%t63)




```
➔       /*plot2d([[discrete, c3], [discrete, c4]],
        [xlabel, "time"],
        [legend, "phi/10", "Z"],
        [gnuplot_preamble, "unset yzeroaxis;"],
        [gnuplot_term, "png linewidth 2.5 font 'Arial' 16 size 800,600"],
        [gnuplot_out_file, "D:/Doc/Artikel-Eck/20-Gyroscope/Fig10.png"])$*/;
```

5.4 Plot space curve of centre of mass

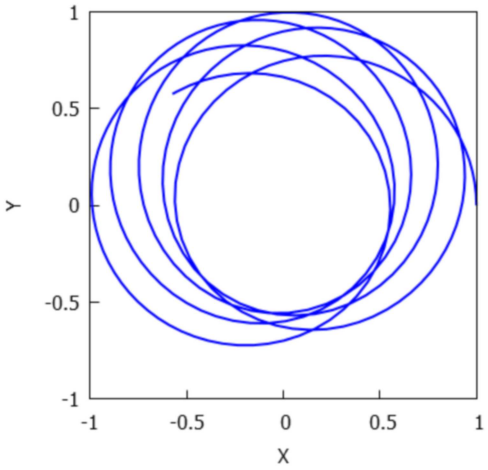
```
(%i64)  transf() := (
            block([i,t,r,phi,x,y,z],
            cl1: [],
            cl2: [],
            for i:1 thru length(c1) step 1 do (
                [t,phi,r]: [c1[i][1],c3[i][2],c4[i][2]],
                x: r*cos(phi),
                y: r*sin(phi),
                z: 0,
                cl1: append(cl1, [[x,y,z]]),
                cl2: append(cl2, [[x,y]])
            )))$
```

```
(%i66)  transf()$
        cr1: points(cl1)$

(%i67)  wxdraw2d(user_preamble="set grid polar; set size square; ",
            xrange=[-3.,3.], yrange=[-3.,3.],
            xrange=[-1.,1.], yrange=[-1.,1.],
            line_width=2, points_joined = true, point_size=0, cr1)$
```

```
 ;;  Loading file C:\Users\test\maxima\binary\5_38_1_5_gdf93b7b_dirty\clisp\2_49__2010_07_07___built_on_toshiba
;;  Loaded file C:\Users\test\maxima\binary\5_38_1_5_gdf93b7b_dirty\clisp\2_49__2010_07_07___built_on_toshiba__
;;  Loading file C:\Users\test\maxima\binary\5_38_1_5_gdf93b7b_dirty\clisp\2_49__2010_07_07___built_on_toshiba_
;;  Loaded file C:\Users\test\maxima\binary\5_38_1_5_gdf93b7b_dirty\clisp\2_49__2010_07_07___built_on_toshiba__
;;  Loading file
C:\Users\test\maxima\binary\5_38_1_5_gdf93b7b_dirty\clisp\2_49__2010_07_07___built_on_toshiba__192_168_43_3__\s
;;  Loaded file C:\Users\test\maxima\binary\5_38_1_5_gdf93b7b_dirty\clisp\2_49__2010_07_07___built_on_toshiba__
;;  Loading file C:\Users\test\maxima\binary\5_38_1_5_gdf93b7b_dirty\clisp\2_49__2010_07_07___built_on_toshiba_
;;  Loaded file C:\Users\test\maxima\binary\5_38_1_5_gdf93b7b_dirty\clisp\2_49__2010_07_07___built_on_toshiba__
0 errors, 0 warnings
```

(%t67)




```
➔       /*draw2d(user_preamble="set grid polar; set size square;
            xrange=[-2.,2.], yrange=[-2.,2.],
            line_width=2, points_joined = true, point_size=0, cr1)$*/
        ;

➔       /*wxplot2d([discrete, cl2],[x,-2,-0], [y,0.5,2],
            [gnuplot_preamble,"set grid polar; "]
             )$*/;
```

5.5 Plot angular momentum

```
(%i68)  Gamma;
```

(%o68)

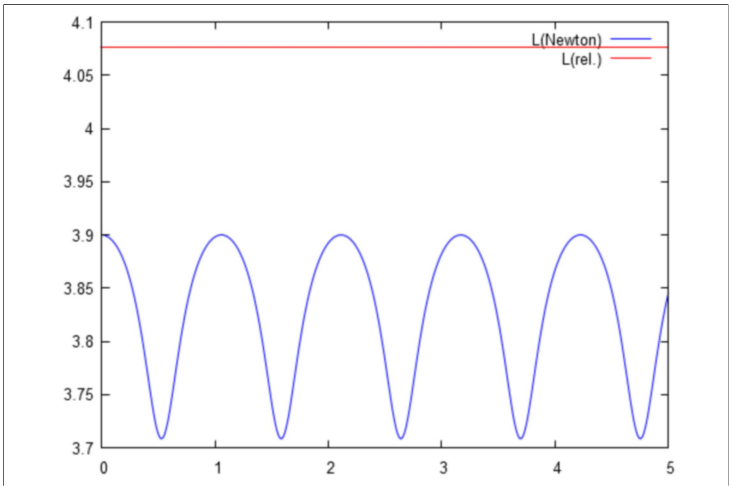
$$\frac{30}{\sqrt{-\frac{r_d^2+\varphi_d^2\,r^2-900\left(-\frac{1}{45\,r}-\frac{0.001}{r^2}+1\right)^2}{-\frac{1}{45\,r}-\frac{0.001}{r^2}+1}}}$$

(%i69)
```
transfc() := (
    block([i,t1,r,phi,r_d,phi_d,x,y,z,m,L,L_N,om1,dom1,m1,
            cc1,gamma,E,E_N,M1,G1],
    cl1: [],cl2: [], cl3: [], cl4: [], cl5: [],
    m1: ev(m,str),
    cc1: ev(c,str),
    M1: ev(M,str),
    G1: ev(G,str),
    om1: subst(t1,t,om[1]),
    dom1: subst(t1,t,dom[1]),
    /*print(om1,dom1),*/
    for i:1 thru length(c1) step 1 do (
        [t1,phi,r,phi_d,r_d]: [c1[i][1],
            c3[i][2],c4[i][2],
            c1[i][2],c2[i][2]],
            /* Newtonian L */
        L_N: m1*r^2*phi_d,
        cl1: append(cl1, [[t1,L_N]]),
            /* gamma*/
        gamma: ev(Gamma),
        cl3: append(cl3, [[t1,gamma]]),
            /* relativistic L */
        L: gamma/ev(m(r))*m1*r^2*phi_d,
        /*L: m1*r^2*(phi_d+ev(om1)+ev(dom1)*t1)/sqrt(1-(r_d^2+r^2*phi_d^2)/cc1^2),*/
        /*L: m1*r^2*(phi_d+ev(om1)+ev(dom1)*t1),*/
        /*L: m1*r^2*(phi_d-ev(om1)-ev(dom1)*t1)
            -2*m1^2*r*(ev(dom1)*t1+ev(om1)),*/
        cl2: append(cl2, [[t1,L]]),
            /* Total anergy */
        E: m1*cc1^2*(ev(m(r))*gamma-1)-sqrt(ev(m(r)))*m1*M1*G1/r,
        cl4: append(cl4, [[t1,E]]),
        E_N: 1/2*m1*(r_d^2+r^2*phi_d^2)-m1*M1*G1/r,
        cl5: append(cl5, [[t1,E_N]])
))$
```

(%i70)
```
transfc()$
/*cr1: points(cl1)$
cr2: points(cl2)$*/
```

(%i71)
```
wxplot2d([[discrete, cl1],[discrete, cl2]],/*[y,0,6],*/
[legend, "L(Newton)", "L(rel.)"])$
```

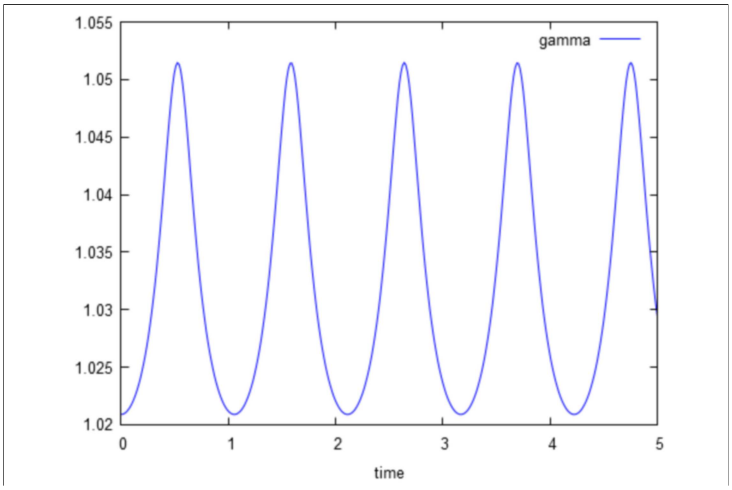
(%t71)



➔
```
/*plot2d([[discrete, cl1], [discrete, cl2]],
[xlabel, "time"],
[legend, "L(Newton)", "L"],
[gnuplot_term, "png linewidth 3 font 'Arial' 16 size 800,600"],
[gnuplot_out_file, "D:/Doc/Artikel-Eck/ECE-Theorie/Paper413/Fig6.png"])$*/;
```

```
(%i72)  wxplot2d([[discrete, cl3]],
        [legend, "gamma"], [xlabel, "time"], [ylabel, ""])$
```

(%t72)




```
(%i73)  wxplot2d([[discrete, cl5], [discrete, cl4]],
        [legend, "E(Newton)", "E(rel.)"], [xlabel, "time"], [ylabel, ""])$
```

(%t73)