144(5): Triple Cross Check with the Hamiltonian Method.

The ECE2 covariant hamiltonian is:

$$H = \gamma m c^2 - \frac{mMG}{r} \quad -(1)$$

The hamiltonian is a constant of motion so:

$$\frac{dH}{dt} = 0 \quad -(2)$$

From eqs. (1) and (2)

$$mc^2 \frac{d\gamma}{dt} = mMG \frac{d}{dt}\left(\frac{1}{r}\right) \quad -(3)$$

Now use:

$$\frac{d}{dt}\left(\frac{1}{r}\right) = -\frac{\dot{r}}{r^2} \quad -(4)$$

where

$$v = \dot{r} \quad -(5)$$

So

$$c^2 \frac{d\gamma}{dt} = -v\frac{MG}{r^2} \quad -(6)$$

The relativistic force is:

$$F = \frac{d}{dt}(\gamma m v) = mv\frac{d\gamma}{dt} + m\gamma\frac{dv}{dt} \quad -(7)$$

So

$$v\frac{d\gamma}{dt} + \gamma\frac{dv}{dt} = -\frac{MG}{r^2} \quad -(8)$$

i.e.

$$-\frac{v^2}{c^2}\frac{MG}{r^2} + \gamma\frac{dv}{dt} = -\frac{MG}{r^2} \quad -(9)$$

and

$$\gamma\frac{dv}{dt} = -\frac{MG}{r^2}\left(1 - \frac{v^2}{c^2}\right) \quad -(10)$$

so

$$\boxed{\gamma^3\frac{dv}{dt} = -\frac{MG}{r^2}} \quad -(11)$$

which is the same result as the kinematic and Lagrangian method.