114(4) : Kinematic and Lagrangian Methods

The kinematic method considers the relativistic velocity:

$$\underline{v} = \gamma \dot{\underline{r}} \quad -(1)$$

where γ is the Lorentz factor. Eq. (1) is true in any coordinate system.

The acceleration is:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\gamma \dot{\underline{r}}\right)$$

$$= \frac{d\gamma}{dt}\dot{\underline{r}} + \gamma \ddot{\underline{r}} \quad -(2)$$

In the plane polar system in the observer frame (r, ϕ):

$$\dot{\underline{r}} = \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi \quad -(3)$$

and

$$\ddot{\underline{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\underline{e}_\phi \quad -(4)$$

Therefore the relativistic acceleration is:

$$\underline{g} = \frac{d\gamma}{dt}\left(\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi\right) + \gamma\left(\left(\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\underline{e}_\phi\right)$$

$$= -\frac{MG}{r^2}\,\underline{e}_r \quad -(5)$$

i.e.

$$\underline{F} = m\underline{g} = -\frac{mMG}{r^2}\,\underline{e}_r, \quad -(6)$$

which is the relativistic equation of orbits.

From eq. (5):

$$\frac{d\gamma}{dt}\dot{r} + \gamma\left(\ddot{r} - r\dot{\phi}^2\right) = -\frac{MG}{r^2} \quad -(7)$$

and

$$\frac{d\gamma}{dt} r\dot{\phi} + \gamma\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) = 0 \quad -(8)$$

Eqs. (7) and (8) can be integrated simultaneously

2) to give the orbit. The Lorentz factor is:

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(9)$$

where

$$v_N^2 = \dot{r}^2 + r^2 \dot{\phi}^2 \quad -(10)$$

Eq. (7) is the relativistic Leibnitz equation and eq. (8) is the equation of conservation of the relativistic angular momentum:

$$\frac{dL}{dt} = 0 \quad -(11)$$

where

$$L = \gamma m r^2 \dot{\phi} \quad -(12)$$

Using the frame transformation:

$$\phi' = \phi + \omega_1 t \quad -(13)$$

Eqs. (7) and (8) become:

$$\frac{d\gamma}{dt}\dot{r} + \gamma\left(\ddot{r} - r\dot{\phi}'^2\right) = -\frac{mG}{r^2} \quad -(14)$$

and

$$\frac{d\gamma}{dt} r\dot{\phi}' + \gamma\left(r\ddot{\phi}' + 2\dot{r}\dot{\phi}'\right) = 0 \quad -(15)$$

and the Lorentz factor becomes:

$$\gamma = \left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\phi}'^2\right)\right)^{-1/2} \quad -(16)$$

Eqs (14) and (15) give the relativistic, shrinking, and precessing orbit, Q.E.D.

The problem has already been solved but the calculation can be checked with the Lagrangian method. The relativistic Lagrangian in frame (r, ϕ) is:

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r} \quad -(17)$$

where

$$\gamma = \left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{-1/2} \quad -(18)$$

Assume that the Lagrange variables are r and ϕ. The Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(19)$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(20)$$

The Lagrangian (17) can be written as:

$$\mathcal{L} = -mc^2 f^{1/2} + \frac{mMG}{r} \quad -(21)$$

where

$$f = \left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right)\right) \quad -(22)$$

so

$$\frac{\partial \mathcal{L}}{\partial f} = -\frac{1}{2}mc^2 f^{-1/2} = -\frac{1}{2}\gamma mc^2. \quad -(23)$$

Therefore

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial \mathcal{L}}{\partial f}\frac{\partial f}{\partial r} + mMG\frac{\partial}{\partial r}\left(\frac{1}{r}\right) \quad -(24)$$

$$\frac{\partial f}{\partial r} = -\frac{2r}{c^2}\dot{\phi}^2, \quad \frac{\partial f}{\partial \dot{r}} = -\frac{2\dot{r}}{c^2} \quad -(25)$$

so

$$\frac{\partial \mathcal{L}}{\partial r} = \gamma m r\dot{\phi}^2 - \frac{mMG}{r^2} \quad -(26)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \gamma m\dot{r} \quad -(27)$$

Therefore:

$$\frac{d}{dt}(\dot{r}\gamma) - \gamma r\dot{\phi}^2 = -\frac{mG}{r^2} \quad -(28)$$

i.e.

$$\frac{d\gamma}{dt}\dot{r} + \gamma(\ddot{r} - r\dot{\phi}^2) = -\frac{mG}{r^2} \quad -(29)$$

which is eq. (7), Q.E.D.

The Lagrangian and kinematic methods give the same results.

Eq. (20) gives:

$$\frac{dL}{dt} = 0 \quad -(30)$$

where

$$L = \gamma m r^2 \dot{\phi} \quad -(31)$$

is the relativistic angular momentum. Therefore:

$$\frac{d}{dt}(\gamma m r^2 \dot{\phi}) = 0 \quad -(32)$$

i.e.

$$r^2\dot{\phi}\frac{d\gamma}{dt} + \gamma\frac{d}{dt}(r^2\dot{\phi}) = 0 \quad -(33)$$

$$= r^2\dot{\phi}\frac{d\gamma}{dt} + \gamma\left(\frac{dr^2}{dt}\dot{\phi} + r^2\frac{d\dot{\phi}}{dt}\right)$$

$$= r^2\dot{\phi}\frac{d\gamma}{dt} + \gamma(2r\dot{r}\dot{\phi} + r^2\ddot{\phi})$$

So:

$$r\dot{\phi}\frac{d\gamma}{dt} + \gamma(2\dot{r}\dot{\phi} + r\ddot{\phi}) = 0 \quad -(34)$$

which is eq. (8) Q.E.D.

5) In order to obtain eqs. (14) and (15) use:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(35)$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi'} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}'} \quad -(36)$$

in which the Lagrange variables are r and ϕ'.

In eq. (7):

$$\dot{r} = v \quad -(37)$$

and

$$\frac{dv}{dt} = \ddot{r} - r\dot{\phi}^2 \quad -(38)$$

so eq. (7) is:

$$F = m\left(v\frac{d\gamma}{dt} + \gamma\frac{dv}{dt}\right) = -\frac{mMG}{r^2} \quad -(39)$$

$$= \frac{d}{dt}(\gamma m v)$$

Write eq. (39) as:

$$F = m\frac{dv}{dt}\left(v\frac{d\gamma}{dv} + \gamma\right) \quad -(40)$$

where we have used:

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} \quad -(41)$$

In eq. (40):

$$\frac{d\gamma}{dv} = \gamma^3 \frac{v}{c^2} \quad -(42)$$

so

$$F = m\gamma\frac{dv}{dt}\left(1 + \gamma^2\frac{v^2}{c^2}\right) \quad -(43)$$

i.e

$$F = m\gamma \frac{dv}{dt}\left(1 + \frac{v^2}{c^2\left(1-\frac{v^2}{c^2}\right)}\right)$$

$$= m\gamma \frac{dv}{dt}\left(\frac{c^2\left(1-\frac{v^2}{c^2}\right)+v^2}{c^2\left(1-\frac{v^2}{c^2}\right)}\right)$$

$$= m\gamma^3 \frac{dv}{dt} \quad -(44)$$

So eq (7) becomes:

$$F = m\gamma^3 \frac{dv}{dt} = -\frac{mMG}{r^2} \quad -(45).$$

which is the relativistic form of Newton's second law, Q.E.D.

It has been shown that

$$\frac{d\gamma}{dt}\dot{r} + \gamma\left(\ddot{r} - r\dot{\phi}^2\right) = \gamma^3 \frac{dv}{dt} \quad -(46)$$

where:

$$v = \dot{r} \quad -(47)$$

and

$$\frac{dv}{dt} = \ddot{r} - r\dot{\phi}^2 \quad -(48)$$

In the frame (r, ϕ')

$$F = m\left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}'^2}{c^2}\right)^{-3/2}\frac{dv}{dt}$$

$$= -mMG/r^2 \quad -(49)$$

) and gives a relativistic, shrinking and precessing orbit when solved simultaneously with:

$$\frac{d}{dt}\left(\gamma m r^2 \dot{\phi}'\right) = 0 \quad -(50)$$

in eq. (50):

$$\gamma = \left(1 - \frac{\dot{r}^2 + r^2 \dot{\phi}'^2}{c^2}\right)^{-1/2} \quad -(51)$$

Here:

$$\dot{\phi}'^2 = \left(\dot{\phi} + \omega_1 + t\frac{d\omega_1}{dt}\right)^2 \quad -(52)$$

The simultaneous equations for numerical integration are therefore:

$$\boxed{\begin{aligned} F &= m\gamma^3 \frac{dv}{dt} = -\frac{mMG}{r^2} \quad -(53) \\ \frac{dL}{dt} &= 0 \quad -(54) \end{aligned}}$$

and

here

$$L = \gamma m r^2 \dot{\phi}'^2 \quad -(54)$$

which the Lorentz factor is:

$$\gamma = \left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\left(\omega + \omega_1 + t\, d\omega_1/dt\right)^2\right)\right)^{-1/2} \quad -(55)$$

If we define:

$$\gamma_0 = \left(1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\omega^2\right)\right)^{-1/2} \quad -(56)$$

) then eq. (53) can be expressed as:

$$F = m \gamma_0^3 \frac{dv}{dt} = -\frac{mMG}{r^2} + \Omega_r \Phi \quad -(57)$$

where (57) is expressed in terms of the spin connection Ω_r and where

$$\Phi = -\frac{MG}{r} \quad -(58)$$

is the gravitational potential.

Computer algebra can be used to find the spin connection Ω_r and the vacuum force:

$$F(\text{vac}) = \Omega_r \Phi \quad -(59)$$

These equations are universal, and can be used to describe any observable orbit.
