414(3): Relativistic Hamiltonian Method

The relativistic Hamiltonian is:

$$H = \gamma m c^2 - \frac{m M G}{r} \quad -(1)$$

and is a constant of motion. So:

$$\frac{dH}{dt} = 0 \quad -(2)$$

In the (r, ϕ') system:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(3)$$

where

$$v^2 = \dot{r}^2 + r^2 \dot{\phi}'^2 \quad -(4)$$

Here:

$$\phi' = \phi + \omega_1 t \quad -(5)$$

It follows that:

$$\frac{d\gamma}{dt} + \frac{MG}{c^2} \frac{\dot{r}}{r^2} = 0 \quad -(6)$$

where we have used:

$$\frac{d}{dt}\left(\frac{1}{r}\right) = \dot{r} \frac{d}{dr}\left(\frac{1}{r}\right) = -\frac{\dot{r}}{r^2} \quad -(7)$$

Define:

$$\gamma = f^{-1/2} \quad -(8)$$

then

$$\frac{d\gamma}{dt} = \frac{d\gamma}{df}\frac{df}{dt} = -\frac{1}{2}\gamma^3 \frac{d}{dt}\left(1 - \frac{\dot{r}^2 + r^2 \dot{\phi}'^2}{c^2}\right)$$

$$= \frac{\gamma^3}{c^2}\left(\dot{r}\ddot{r} + r\dot{\phi}'^2 \dot{r} + r^2 \dot{\phi}' \ddot{\phi}'\right) \quad -(8)$$

From eqs. (6) and (8):

$$\gamma^3\left(\dot{r}\ddot{r} + r\dot{\phi}'^2\dot{r} + r^2\dot{\phi}'\ddot{\phi}'\right) = -\frac{mG}{r^2}\dot{r} \quad -(9)$$

i.e $$\gamma^3\left(\dot{r}\ddot{r} - r\dot{\phi}'^2\dot{r} + 2r\dot{\phi}'^2 + r^2\dot{\phi}'\ddot{\phi}'\right) = -\frac{mG}{r^2}\dot{r} \quad -(10)$$

Possible solutions of eq. (10) are:

$$\boxed{\begin{aligned}\gamma^3\left(\ddot{r} - r\dot{\phi}'^2\right) &= -\frac{mG}{r^2} \quad -(11)\\ \gamma^3\left(2\dot{\phi}'\dot{r} + r\ddot{\phi}'\right) &= 0 \quad -(12)\end{aligned}}$$

and

These are the fully relativistic equations of a precessing and shrinking orbit.

They can be written as:

$$\ddot{r} - r\dot{\phi}'^2 = -\frac{mG}{\gamma^3 r^2} \quad -(13)$$

and

$$2\dot{\phi}'\dot{r} + r\ddot{\phi}' = 0 \quad -(14)$$

sing:

$$\dot{\phi}'^2 = \left(\dot{\phi} + \omega_1 + t\frac{d\omega_1}{dt}\right)^2 \quad -(15)$$

transforms eqs. (13) and (14) into the observer frame
r, ϕ) giving:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{mG}{\gamma^3 r^2} + \Omega_r r \quad -(16)$$

$$2\dot{r}(\dot{\phi} + \omega_1 + t\dot{\omega}_1) + r(\ddot{\phi} + 2\dot{\omega}_1 + t\ddot{\omega}_1) = 0 \quad -(17)$$

) where:

$$\Omega_r = -\frac{L}{mMG}\frac{(\omega_1 + \dot{\omega}_1)(\omega_1 + t\dot{\omega}_1 + 2\omega)}{(\omega + \omega_1 + t\dot{\omega}_1)} \quad -(18)$$

is the spin connection.

The calculations of Note 414(2) are recovered in the limit:

$$\gamma \rightarrow 1. \quad -(19)$$

The usual non-relativistic hamiltonian is recovered from eq. (1) using:

$$H_0 = H - mc^2 = (\gamma - 1)mc^2 - \frac{mMG}{r} \quad -(20)$$

where:

$$T = (\gamma - 1)mc^2 \quad -(21)$$

is the relativistic kinetic energy. Using:

$$T = \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)mc^2 \quad -(22)$$

$$\sim \left(1 + \frac{v^2}{c^2} + \dots - 1\right)mc^2$$

$$= \frac{1}{2}mv^2$$

gives the non-relativistic kinetic energy when:

$$v \ll c \quad -(23)$$

Simultaneous numerical solution of eqs. (16) and (17) gives the fully relativistic, shrinking and precessing orbit.