414(1): Summary of the Equations with Spin Connection

These are:

$$F = m\ddot{r} - mr\dot{\phi}^2 = -\frac{mMG}{r^2} + \Omega_r \Phi \quad -(1)$$

and

$$2\dot{\phi}'\dot{r} + r\ddot{\phi}' = 0 \quad -(2)$$

also

$$\phi' = \phi + \omega_1 t. \quad -(3)$$

The spin connection is:

$$\Omega_r = -\frac{r^2}{MG}\left(\omega_1 + t\frac{d\omega_1}{dt}\right)\left(\omega_1 + t\frac{d\omega_1}{dt} + 2\omega\right) \quad -(4)$$

in which

$$\omega = \frac{d\phi}{dt} \quad -(5).$$

Eq. (2) is equivalent to:

$$\frac{dL}{dt} = 0 \quad -(6)$$

where

$$L = mr^2\left(\omega + \omega_1 + t\frac{d\omega_1}{dt}\right) \quad -(7)$$

So the spin connection is:

$$\Omega_r = -\frac{L}{mMG}\frac{\left(\omega_1 + t\frac{d\omega_1}{dt}\right)\left(\omega_1 + t\frac{d\omega_1}{dt} + 2\omega\right)}{\left(\omega + \omega_1 + t\frac{d\omega_1}{dt}\right)} \quad -(8)$$

Here

$$\Phi = -\frac{MG}{r} \quad -(9)$$

2) In eq. (2):

$$\dot{\phi}' = \dot{\phi} + \omega_1 + t\frac{d\omega_1}{dt} \quad -(10)$$

and

$$\ddot{\phi}' = \ddot{\phi} + \dot{\omega}_1 + \dot{\omega}_1 + t\ddot{\omega}_1 \quad -(11)$$

$$= \ddot{\phi} + 2\dot{\omega}_1 + t\ddot{\omega}_1$$

Therefore eq. (2) is:

$$2\dot{r}\left(\dot{\phi} + \omega_1 + t\dot{\omega}_1\right) + r\left(\ddot{\phi} + 2\dot{\omega}_1 + t\ddot{\omega}_1\right) = 0 \quad -(12)$$

Therefore eqs. (1) and (12) must be solved simultaneously, with the spin connection (8).

By definition:

$$\dot{\phi}'^2 = \left(\dot{\phi} + \omega_1 + t\frac{d\omega_1}{dt}\right)^2 \quad -(13)$$

$$= \dot{\phi}^2 + 2\left(\omega_1 + t\frac{d\omega_1}{dt}\right)\dot{\phi} + \left(\omega_1 + t\frac{d\omega_1}{dt}\right)^2$$

$$= \dot{\phi}^2 - \frac{MG}{r^2}\Omega_r \quad -(14)$$

The hamiltonian is:

$$H = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}'^2\right) - \frac{mMG}{r}$$

So:

$$\frac{dH}{dt} \underset{\omega_1 \to 0}{=} 0 \quad -(15)$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) + \frac{mMG}{r} \quad -(16)$$

The orbit from eq. (14) is:

$$r = \frac{\alpha}{1 + \epsilon \cos(\phi + \omega_1 t)} \quad -(17)$$

from analysis of the hamiltonian. Together with eq. (17)

$$L = m r^2 \left(\omega + \omega_1 + t \frac{d\omega_1}{dt} \right) \quad -(18)$$

from considerations of the Lagrangian. Therefore:

$$r^2 = \frac{L}{m\left(\omega + \omega_1 + t \frac{d\omega_1}{dt}\right)} \quad -(19)$$

$$= \left(\frac{\alpha}{1 + \epsilon \cos(\phi + \omega_1 t)} \right)^2$$

with:

$$\frac{dL}{dt} = 0 \quad -(20)$$

The semi major axis is

$$a = \frac{\alpha}{1 - \epsilon^2} = \frac{mMG}{2|H|} \quad -(21)$$

and the semi minor axis is:

$$b = (\alpha a)^{1/2} = \frac{\alpha}{(1-\epsilon^2)^{1/2}} \quad -(22)$$

It is clear that:

$$a \xrightarrow[\omega_1 \to \infty]{} 0 \quad -(23)$$

$$b \xrightarrow[\omega_1 \to \infty]{} 0 \quad -(24)$$

) In eq. (17):

$$\alpha = \frac{L^2}{m^2 M G} \quad -(25)$$

and

$$\epsilon = \left(1 + \frac{2HL^2}{m^3 M^2 G^2} \right)^{1/2} \quad -(26)$$

Solution of eqs. (1) and (12) should give eq. (17).

In the Newtonian limit:

$$\Omega_r = 0 \quad -(27)$$

so

$$\phi' = \phi \quad -(28)$$

and eqs. (1) and (2) become:

$$F_N = m\ddot{r} - mr\dot{\phi}^2 = -\frac{mMG}{r^2} \quad -(29)$$

and

$$2\dot{\phi}\dot{r} + r\ddot{\phi} = 0 \quad -(30)$$

i.e.

$$L_N = mr^2\omega \quad -(31)$$

and

$$\frac{dL_N}{dt} = 0 \quad -(32)$$

Integrating eqs (29) and (30) numerically should give

$$r = \frac{\alpha_N}{1 + \epsilon_N \cos\phi} \quad -(33)$$

In the Newtonian limit:

$$r^2 = \frac{L_N}{m\omega} = \left(\frac{d_N}{1+\epsilon_N \cos\phi}\right)^2 \quad - (34)$$

For a given ω, the orbit is stable (an ellipse), if:

$$L_N = mr^2\omega = \text{constant} \quad - (35)$$

i.e

$$\frac{dL_N}{dt} = 0. \quad - (36)$$

In a pure mathematical sense:

$$r \xrightarrow[\omega \to \infty]{} 0. \quad - (37)$$

An orbit with a very large ω means a very small r, because L_N must be constant. The difference between eqs. (19) and (34) is that in Eq. (19), the time t appears, as a result of the coordinate change:

$$\phi' = \phi + \omega_1 t, \quad - (38)$$

So:

$$r \xrightarrow[t \to \infty]{} 0 \quad - (39)$$

for a given ω_1 and $d\omega_1/dt$, in order to keep L a constant:

$$\frac{dL}{dt} = 0 \quad - (40)$$

In the fully relativistic theory:

$$H = \gamma mc^2 - mMG/r \quad - (41)$$

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r} \quad - (42)$$

so interesting new effects appear.