43(8): Transformation of Coordinate Proof of Orbital Precession and Shrinking.

The plane polar coordinates are transformed from (r, ϕ) to (r, ϕ'), where:

$$\phi' = \phi + \omega_1 t \quad - (1)$$

It follows that the hamiltonian is transformed to

$$H = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}'^2 \right) - \frac{mMG}{r} \quad - (2)$$

and that the Lagrangian is transformed to:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}'^2 \right) + \frac{mMG}{r} \quad - (3)$$

From eq. (2):

$$\phi'(r) = \int \frac{L}{r^2} \frac{dr}{\left(2\mu \left(H - U - \frac{L^2}{2\mu r^2} \right) \right)^{1/2}} \quad - (4)$$

so

$$\cos \phi' = \frac{\frac{L^2}{\mu k r} - 1}{\left(1 + \frac{2HL^2}{\mu k^2} \right)^{1/2}} \quad - (5)$$

where

$$L = \mu r^2 \dot{\phi}' \quad - (6)$$

Eq. (5) is the conic section:

$$r = \frac{\alpha}{1 + \epsilon \cos \phi'} \quad - (7)$$

which is a precessing ellipse.

) In these equations the hamiltonian H and total angular momentum L are constants of motion, so:

$$\frac{dH}{dt} = 0, \quad \frac{dL}{dt} = 0. \quad -(8)$$

The half right latitude is:

$$\alpha = \frac{L^2}{\mu k} \quad -(9)$$

and the ellipticity is:

$$\epsilon = \left(1 + \frac{2HL^2}{\mu k^2} \right)^{1/2} \quad -(10)$$

where the reduced mass is:

$$\mu = \frac{mM}{m+M} \quad -(11)$$

and

$$k := mMG \quad -(12)$$

If T is the time to complete a revolution of 2π, then the precession is:

$$\Delta\phi = \omega_1 T \quad -(13)$$

This explains the precession of the Hulse Taylor binary pulsar and the S2 star system.

The semi major axis of the orbit is:

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2|H|} \quad -(14)$$

the semi minor axis is:

$$b = \frac{\alpha}{(1-\epsilon^2)^{1/2}} = \frac{L}{(2\mu|H|)^{1/2}} \quad -(15)$$

3) From eqs. (1) and (2):

$$\phi' \xrightarrow[t \to \infty]{} \infty \quad -(16)$$

So:

$$H \xrightarrow[t \to \infty]{} \infty \quad -(17)$$

In consequence:

$$a \xrightarrow[t \to \infty]{} 0 \quad -(18)$$

and the orbit shrinks to zero, Q.E.D.

Kepler's Law for areal velocity (Kepler's second law) is:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi}' = \frac{L}{2\mu} = \text{constant} \quad -(19)$$

so

$$\boxed{r \xrightarrow[\dot{\phi}' \to \infty]{} 0} \quad -(20)$$

Q.E.D. This explains the shrinking of the orbit of the Hulse Taylor binary pulsar without any gravitational radiation, Q.E.D.

Kepler's first law is modified to Eq. (7), to give a precessing orbit.

Kepler's third law is the direct result of eq. (19):

$$dt = \frac{2\mu}{L} dA \quad -(21)$$

So:

4)

$$T = \int_0^t dt = \frac{2\mu}{L}\int_0^A dA \quad -(22)$$

so

$$T = \frac{2\mu}{L} A \quad -(23)$$

The area of the ellipse is:

$$A = \pi a b \quad -(24)$$

Here

$$b = (\alpha a)^{1/2} \quad -(25)$$

so from Eqs. (9), (14) and (15):

$$T^2 = \frac{4\pi^2 \mu}{k} a^3 \quad -(26)$$

so

$$\boxed{\begin{array}{c} T \longrightarrow 0 \\ \dot{\phi} \to \infty \end{array}} \quad -(27)$$

Eq. (26) is Kepler's third law.