H3(5): Derivation of the Spin Connection From Frame Rotation

The frame rotation is derived from the new coordinate system:

$$\phi' = \phi + \omega_1 t \quad -(1)$$

The new frame of reference is $(r, \phi')$. In this new frame the hamiltonian is

$$H = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\phi}'^2 \right) - \frac{mMG}{r} \quad -(2)$$

As shown in the previous note:

$$\frac{dH}{dt} = 0 \quad -(3)$$

gives:

$$m\ddot{r} - mr\dot{\phi}^{2\prime} = -\frac{mMG}{r^2} \quad -(4)$$

and

$$2\dot{\phi}'\dot{r} + r\ddot{\phi}' = 0 \quad -(5).$$

The orbit from eq. (2) is:

$$r = \frac{\alpha}{1 + \epsilon \cos \phi'} \quad -(6)$$

$$= \frac{\alpha}{1 + \epsilon \cos(\phi + \omega_1 t)}$$

which is a precessing ellipse. The precession at the perihelion is:

$$\Delta\phi = \omega_1 T \quad -(7)$$

where $T$ is the time needed to complete an orbit of $2\pi$ radians.

In eq. (4):

$$\dot{\phi}'^2 = \left( \omega + \omega_1 + t \frac{d\omega_1}{dt} \right)^2 \quad -(8)$$

So
$$\dot{\phi}'^2 = \dot{\phi}^2 + 2\left(\omega_1 + t\frac{d\omega_1}{dt}\right)\dot{\phi} + \left(\omega_1 + t\frac{d\omega_1}{dt}\right)^2 \quad -(9)$$

Eq. (4) is therefore:
$$F = m\ddot{r} - mr\dot{\phi}^2 = -\frac{mMG}{r^2} + m\Omega_r\Phi \quad -(10)$$

where
$$\Phi = -\frac{MG}{r} \quad -(11)$$

and $\Omega_r$ is the radial part of the spin connection.

Therefore:
$$\Omega_r\Phi = r\left(\omega_1 + t\frac{d\omega_1}{dt}\right)\left(\omega_1 + t\frac{d\omega_1}{dt} + 2\omega\right) \quad -(12)$$

It has been shown that the spin connection $\Omega_r$ is the direct result of the frame rotation (1), Q.E.D. The spacetime torsion is defined by the spin connection through the first Maurer Cartan structure equation, so the cause of frame rotation is the spacetime torsion.

The conserved angular momentum $L$ in the frame $(r, \phi')$ is:
$$L = mr^2\dot{\phi}' \quad -(13)$$
$$= mr^2\left(\omega + \omega_1 + t\frac{d\omega_1}{dt}\right)$$

So
$$r^2 = \frac{L}{m}\left(\omega + \omega_1 + t\frac{d\omega_1}{dt}\right) \quad -(14)$$

) It follows that:

$$\Omega_r = -\frac{r^2}{mG}\left(\omega_1 + t\frac{d\omega_1}{dt}\right)\left(\omega_1 + t\frac{d\omega_1}{dt} + 2\omega\right)$$

$$\boxed{\Omega_r = -\frac{L}{mMG}\left(\omega + \omega_1 + t\frac{d\omega_1}{dt}\right)\left(\omega_1 + t\frac{d\omega_1}{dt}\right)\left(\omega_1 + t\frac{d\omega_1}{dt} + 2\omega\right)} \quad -(15)$$

Therefore the spin connection can be expressed in terms of $\omega$, $\omega_1$, $\frac{d\omega_1}{dt}$ and $t$. If there is no frame rotation the spin connection is self consistently zero.