413(1): Expression for Orbital Shrinkage

In the rotating frame, the constant angular momentum is:

$$L' = mr^2\left(\omega + \omega_1 + t\frac{d\omega_1}{dt}\right) \quad -(1)$$

So:

$$r^2 = \frac{L'}{m\left(\omega + \omega_1 + t\frac{d\omega_1}{dt}\right)} \quad -(2)$$

and therefore with increasing time, the orbital radius r must decrease and the orbit must shrink. Orbital shrinkage is observed astronomically in binary pulsars. The shrinkage occurs when

$$\frac{d\omega_1}{dt} \neq 0 \quad -(3)$$

so that ω_1 is time dependent and

$$\omega_1 = \omega_1(t) \quad -(4)$$

If ω_1 is constant the orbit does not shrink. The rotating frame needed to define orbital shrinkage is therefore:

$$\phi' = \phi + t\,\omega_1(t) \quad -(5)$$

The observed rate of shrinkage can be calculated from:

$$\frac{dL'}{dt} = 0 \quad -(6)$$

so:

$$\frac{d}{dt}\left(r^2\left(\omega + \omega_1 + t\frac{d\omega_1}{dt}\right)\right) = 0. \quad -(7)$$

Using the Leibniz Theorem:

$$\left(\omega+\omega_1+t\frac{d\omega_1}{dt}\right)\frac{d}{dt}(r^2)+r^2\frac{d}{dt}\left(\omega+\omega_1+t\frac{d\omega_1}{dt}\right)=0 \quad -(8)$$

in which

$$\frac{d}{dt}(r^2)=\frac{d}{dr}(r^2)\frac{dr}{dt} \quad -(9)$$
$$=2r\frac{dr}{dt}$$

so

$$\frac{dr}{dt}=-\frac{r}{2}\frac{\frac{d}{dt}\left(\omega+\omega_1+t\frac{d\omega_1}{dt}\right)}{\omega+\omega_1+t\frac{d\omega_1}{dt}} \quad -(10)$$

in which the orbital angular velocity is:

$$\frac{d\phi}{dt}=\omega=\frac{2\pi}{T}. \quad -(11)$$

The frame rotation is defined by

$$\phi'=\phi+\omega_1 t \quad -(12)$$

so in eq. (11) T is the time needed to complete an orbit of 2π radians in the unrotated frame. By Kepler's third law:

$$T^2=\frac{4\pi^2}{MG}a^3 \quad -(13)$$

where a is the semi major axis of the unrotated frame, so T is a constant of motion because a is a constant of motion. It follows that:

$$\frac{d\omega}{dt}=0 \quad -(14)$$

So
$$\frac{dr}{dt} = -\frac{r}{2}\,\frac{\frac{d}{dt}\left(\omega_1 + t\frac{d\omega_1}{dt}\right)}{\omega + \omega_1 + t\frac{d\omega_1}{dt}} \quad -(15)$$

$$= -\frac{r}{2}\,\frac{\left(\frac{d\omega_1}{dt} + \frac{d\omega_1}{dt} + t\frac{d^2\omega_1}{dt^2}\right)}{\omega + \omega_1 + t\frac{d\omega_1}{dt}}$$

$$= -\frac{r}{2}\,\frac{\left(2\frac{d\omega_1}{dt} + t\frac{d^2\omega_1}{dt^2}\right)}{\omega + \omega_1 + t\frac{d\omega_1}{dt}}$$

By astronomical observation r in a shrinking orbit decreases with time, so
$$\frac{dr}{dt} < 0. \quad -(16)$$

The model:
$$\omega_1 = \omega_1(0)\, e^{t/\tau} \quad -(17)$$

gives the result (15) because:
$$\frac{d\omega_1}{dt} = \frac{\omega_1(0)}{\tau}\, e^{t/\tau} = \frac{\omega_1}{\tau^2} \quad -(18)$$

and
$$\frac{d^2\omega_1}{dt^2} = \frac{\omega_1}{\tau^2} \quad -(19)$$

It follows that:
$$\frac{dr}{dt} = -\frac{r}{2}\,\frac{\left(2\frac{\omega_1}{\tau} + t\frac{\omega_1}{\tau^2}\right)}{\omega + \omega_1 + t\frac{d\omega_1}{dt}} \quad -(20)$$

+) Therefore in the model (17):

$$\frac{dr}{dt} = -\frac{r}{2}\frac{\left(\frac{\omega_1}{\tau}\right)\left(2+\frac{t}{\tau}\right)}{\omega+\omega_1+\frac{t\omega_1}{\tau}} \quad -(21)$$

< 0.

There is another equation available for dr/dt. This is derived from the rotating frame orbit:

$$r = \frac{\alpha'}{1+\epsilon'\cos\phi'} \quad -(20)$$

so

$$\frac{dr}{dt} = \frac{dr}{d\phi'}\frac{d\phi'}{dt} = \frac{\epsilon'}{\alpha'} r^2 \sin\phi' \frac{d\phi'}{dt} \quad -(21)$$

in which ϵ'/α' is a constant of motion.

In eq. (21):

$$\frac{d\phi'}{dt} = \omega + \omega_1\left(1+\frac{t}{\tau}\right) \quad -(22)$$

using the model (17).

It follows that:

$$\frac{dr}{dt} = \frac{\epsilon'}{\alpha'} r^2 \left(\omega+\omega_1\left(1+\frac{t}{\tau}\right)\right)\sin\phi' \quad -(23)$$

Comparing eqs (20) and (23) is possible but it is easier to use:

5)

$$\frac{dr}{dt} = -Ar \quad -(21)$$

where

$$A = \frac{\frac{\omega_1}{2\tau}\left(2+\frac{t}{\tau}\right)}{\omega+\omega_1\left(1+\frac{t}{\tau}\right)} \quad -(22)$$

and

$$\frac{dr}{dt} = \frac{dr}{d\phi'}\frac{d\phi'}{dt} = \frac{L'}{mr^2}\frac{dr}{d\phi'} \quad -(23)$$

with

$$\frac{dr}{d\phi'} = \frac{\epsilon'}{\alpha'} r^2 \sin\phi' \quad -(24)$$

Therefore

$$\frac{dr}{dt} = \frac{L'\epsilon'}{m\alpha'}\sin\phi' = -Ar \quad -(25)$$

in which:

$$\phi' = \phi + \omega_1(0)e^{t/\tau}t \quad -(26)$$

and

$$r = \frac{\alpha'}{1+\epsilon'\cos\phi'} \quad -(27)$$

It follows that:

$$\cos\phi' = \frac{1}{\epsilon'}\left(\frac{\alpha'}{r}-1\right) \quad -(28)$$

and

$$\sin^2\phi' = 1-\cos^2\phi' \quad -(29)$$

so:

$$\sin^2\phi' = 1-\frac{1}{\epsilon'^2}\left(\frac{\alpha'}{r}-1\right)^2 \quad -(30)$$

6) Fm eq. (25):

$$\left(\frac{L' \epsilon'}{m \alpha'}\right)^2 \sin^2 \phi' = A^2 r^2 \quad -(31)$$

So:

$$\left(\frac{m \alpha'}{L' \epsilon'}\right)^2 A^2 r^2 = 1 - \frac{1}{\epsilon'^2}\left(\frac{\alpha'}{r} - 1\right)^2 \quad -(32)$$

Eq. (32) gives an expression for r in terms of A, and therefore in terms of time. It is a quartic equation in r with four roots. The quantities m, α', L', ϵ' are constants, and ω_1 and τ can be adjusted.
