HS(4): Lagrangian in the Observer Frame

In the observer frame:

$$\phi = \phi' - \omega_1 t \quad -(1)$$

and the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{mMG}{r} \quad -(2)$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2(\dot{\phi}' - \omega_1 - t\dot{\omega}_1)^2\right) + \frac{mMG}{r}$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\left(\dot{\phi}'^2 - 2(\omega_1 + t\dot{\omega}_1)\dot{\phi}' + (\omega_1 + t\dot{\omega}_1)^2\right)\right) + \frac{mMG}{r}$$

The Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(3)$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi'} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}'} \quad -(4)$$

Eq (3) gives the Leibniz equation modified by frame rotation:

$$m\ddot{r} - mr(\dot{\phi}' - \omega_1 - t\dot{\omega}_1)^2 = -\frac{mMG}{r^2} \quad -(5)$$

and eq. (4) gives:

$$\frac{dL}{dt} = 0 \quad -(6)$$

where

$$L = mr^2(\dot{\phi}' - \omega_1 - t\dot{\omega}_1) \quad -(7)$$

the angular momentum modified by frame rotation.

) The constants of motion in the presence of frame rotation are L and the hamiltonian:

$$H = \frac{1}{2}m\left(\dot{r}^2 + r^2\left(\dot{\phi}' - \omega_1 - t\dot{\omega}_1\right)^2\right) - \frac{mMG}{r} \quad -(8)$$

where

$$\frac{dH}{dt} = 0 \,. \quad -(9)$$

Eq. (5) is the same as given in the protocol, Q.E.D.

From eqs. (6) and (7):

$$\frac{d}{dt}\left(r^2\left(\dot{\phi}' - \omega_1 - t\dot{\omega}_1\right)\right) = 0 \quad -(10)$$

i.e.

$$\left(\dot{\phi}' - \omega_1 - t\dot{\omega}_1\right)\frac{d}{dt}r^2 + r^2\left(\ddot{\phi}' - \dot{\omega}_1 - \dot{\omega}_1 - t\ddot{\omega}_1\right) = 0 \quad -(11)$$

in which:

$$\frac{d}{dt}r^2 = \frac{dr^2}{dr}\frac{dr}{dt} = 2r\frac{dr}{dt} \,. \quad -(12)$$

Therefore:

$$2\left(\dot{\phi}' - \omega_1 - t\dot{\omega}_1\right)\dot{r} + r\left(\ddot{\phi}' - 2\dot{\omega}_1 - t\ddot{\omega}_1\right) = 0 \quad -(13)$$

and

$$m\ddot{r} - mr\left(\dot{\phi}' - \omega_1 - t\dot{\omega}_1\right)^2 = -\frac{mMG}{r^2} \quad -(14)$$

If there is no frame rotation:

$$\omega_1 = 0 \quad -(15)$$

and

$$\phi' = \phi \quad -(16)$$

so

$$2\dot{\phi}\dot{r} + r\ddot{\phi} = 0 \quad -(17)$$

and

$$m\ddot{r} - mr\dot{\phi} = -\frac{mMG}{r^2} \quad -(18)$$

which is the correct result for Newtonian dynamics Q.E.D.

) Eqs. (17) and (18) give the Newtonian conic section:

$$r = \frac{\alpha}{1+\epsilon\cos\phi}, \quad -(19)$$

for example the static ellipse.

The protocol solves eqs. (13) and (14) to give the dynamics of r and ϕ' in a precessing orbit, and the observer frame. These are very original and highly interesting results by coauthor Horst Eckardt.

In addition to the information given by the protocol, there is additional information contained in eq. (9):

$$\frac{1}{2}\frac{d}{dt}\left(\dot{r}^2 + r^2(\dot{\phi}' - \omega_1 - t\dot{\omega}_1)^2\right) = \frac{d}{dt}\left(\frac{MG}{r}\right) \quad -(20)$$

In order to evaluate eq. (20) consider first the hamiltonian when there is no frame rotation:

$$H = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{mMG}{r} \quad -(21)$$

Then:

$$\frac{dH}{dt} = 0 = \frac{1}{2}m\left(\frac{d}{dt}\dot{r}^2 + \frac{d}{dt}\left(r^2\dot{\phi}^2\right)\right) - mMG\frac{d}{dt}\left(\frac{1}{r}\right)$$

$$= \frac{1}{2}\left(2\dot{r}\ddot{r} + \frac{dr^2}{dt}\dot{\phi}^2 + r^2\frac{d\dot{\phi}^2}{dt}\right) + \frac{MG}{r^2}\dot{r}$$

Therefore:

$$\dot{r}\ddot{r} + r\dot{\phi}^2\dot{r} + r^2\dot{\phi}\ddot{\phi} + \frac{MG}{r^2}\dot{r} = 0 \quad -(22)$$

Eq. (22) can be written as:

$$r\ddot{r} - r\dot{\phi}^2\dot{r} + 2r\dot{\phi}^2\dot{r} + r^2\dot{\phi}\ddot{\phi} = -\frac{mG}{r^2}\dot{r} \quad -(23)$$

A solution of eq. (23) is:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{MG}{r^2} \quad -(24)$$

and

$$2r\dot{\phi}^2\dot{r} + r^2\dot{\phi}\ddot{\phi} = 0 \quad -(25)$$

i.e.

$$2\dot{\phi}\dot{r} + r\ddot{\phi} = 0 \quad -(26)$$

Eqs. (24) and (26) are the same as eqs. (17) and (18), Q.E.D. Therefore the Hamiltonian and Lagrangian methods give the same result. However, the Lagrangian method has the advantage of giving the constant angular momentum (7).

In order to extend the Hamiltonian analysis to the rotating frame use

$$\dot{\phi}^2 \rightarrow (\dot{\phi}' - \omega_1 - t\dot{\omega}_1)^2 \quad -(27)$$

So eq. (24) gives eq. (14) self consistently, Q.E.D.

Finally use:

$$\ddot{\phi} \rightarrow \ddot{\phi}' - 2\dot{\omega}_1 - t\ddot{\omega}_1 \quad -(28)$$

So eq. (26) gives eq. (13) self consistently, Q.E.D.

In these calculations the plane polar angle of the observer frame is defined by eq. (1):

$$\phi = \phi' - \omega_1 t \quad -(29)$$

and the plane polar angle of the rotating frame is

$$\phi' = \phi + \omega_1 t \quad -(30)$$

) The hamiltonian of the observer frame is:

$$H(\text{obs}) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{mMG}{r} \quad -(31)$$

giving the orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(32)$$

$$= \frac{\alpha}{1+\epsilon\cos(\phi' - \omega_1 t)} \quad -(32a)$$

This is a precessing orbit in the observer frame. After an orbit of 2π radians, the perihelion is displaced by

$$\Delta\phi' = -\omega_1 t. \quad -(33)$$

This is the precession of the perihelion in the observer frame, defined by plane polar coordinates (r, ϕ).

The hamiltonian in the rotating frame is:

$$H(\text{rotating}) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}'^2) - \frac{mMG}{r} \quad -(34)$$

giving the orbit:

$$r = \frac{\alpha'}{1+\epsilon'\cos\phi'}$$

$$= \frac{\alpha'}{1+\epsilon'\cos(\phi + \omega_1 t)} \quad -(35)$$

This is a precessing orbit in the rotating frame. After an orbit of 2π radians, the perihelion is displaced by:

$$\Delta\phi = \omega_1 t \quad -(36)$$

This is the precession of the perihelion in the rotating frame defined by the plane polar coordinates (r, ϕ').

The precessions in the observer and rotating frames are equal and opposite in sign.