410(2): Relation Between Precession, Time Dilatation and Length Contraction

Consider the Lorentz transformation along axis 1:

$$x_1' = \gamma(x_1 - v_N t) \quad -(1)$$

$$x_2' = x_2$$

$$x_3' = x_3$$

$$t' = \gamma\left(t - \frac{v_N x_1}{c^2}\right)$$

where

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(2)$$

is the Lorentz factor. The primed frame moves at a constant velocity v_N with regard to the unprimed frame. Here v_N is the magnitude of the Newtonian velocity.

Consider a rod of length ℓ in the x_1 axis of an inertial frame K. An observer in the K' system moving with uniform velocity v along the x_1 axis measures the length of the rod in the observer's own coordinate system by determining at a given instant t' the difference in the coordinates of the ends of the rod, $x_1'(2) - x_1'(1)$.

From eq. (1):

$$x_1'(2) - x_1'(1) = \gamma\left(x_1(2) - x_1(1) - v\left(t(2) - t(1)\right)\right) \quad -(3)$$

where

$$\ell = x_1(2) - x_1(1) \quad -(4)$$

Using

$$t'(2) = t'(1) \quad -(5)$$

it follows that:

$$t(2) - t(1) = (x_1(2) - x_1(1))\frac{v}{c^2} \quad -(6)$$

In the K′ system:

$$l' = x_1'(2) - x_1'(1) \quad -(7)$$

so eq. (3) becomes:

$$l' = l\left(1 - \frac{v_N^2}{c^2}\right) = \frac{l}{\gamma} \quad -(8)$$

To an observer in motion relative to an object, the dimensions of the object are contracted by a factor $1/\gamma$ in the direction of motion.

This was proposed by G. F. Fitzgerald in 1892 to account for the Michelson Morley experiment, and was immediately developed by H. A. Lorentz and Oliver Heaviside.

Time dilation follows immediately from the infinitesimal line element:

$$ds^2 = c^2 d\tau^2 = (c^2 - v_N^2)\,dt^2 \quad -(9)$$

where $d\tau$ is the time in the moving frame, known as the proper time. It follows from eq. (9) that:

$$dt = \gamma\, d\tau \quad -(10)$$

To an observer in motion relative to an object, the interval of time dt is longer than the interval of proper time $d\tau$ in the frame in which the observer and clock are at rest. To an observer in motion relative to the clock, the time intervals appear to be lengthened, and clocks appear to be slower.

) It follows that:

$$\gamma = \frac{l}{l'} = \frac{\Delta t}{\Delta \tau} \quad -(11)$$

A precession may be defined as: $-(12)$

$$\boxed{\Delta\phi = 2\pi\left(\frac{l}{l'} - 1\right) = 2\pi\left(\frac{\Delta t}{\Delta\tau} - 1\right) = 2\pi(\gamma - 1)}$$

where

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(13)$$

is the Lorentz factor.

Now check the above calculations by considering the inverse Lorentz transform:

$$\begin{aligned} x_1 &= \gamma(x_1' + vt') \\ x_2 &= x_2' \\ x_3 &= x_3' \\ t &= \gamma\left(t' + \frac{v}{c^2}x_1'\right) \end{aligned} \quad -(14)$$

Using: $-(15)$

$$l = x_1(2) - x_1(1) = \gamma\left(x_1'(2) - x_1'(1) + v\left(t'(2) - t'(1)\right)\right)$$

and

$$t(2) - t(1) = \gamma\left(t'(2) - t'(1) + \frac{v}{c^2}\left(x_1'(2) - x_1'(1)\right)\right)$$

$$= 0 \quad -(16)$$

it is found that:

4)

$$x_1(2) - x_1(1) = \gamma\left(1 - \frac{v^2}{c^2}\right)\left(x_1'(2) - x_1'(1)\right) \quad -(17)$$

$$= \left(1 - \frac{v^2}{c^2}\right)^{1/2}\left(x_1'(2) - x_1'(1)\right)$$

i.e.

$$\boxed{\ell = \frac{\ell'}{\gamma}} \quad -(18)$$

This is the inverse of the result (8)

Using eqs. (14) consider: $\quad -(19)$

$$t(2) - t(1) = \gamma\left(t'(2) - t'(1) + \frac{v}{c^2}\left(x_1'(2) - x_1'(1)\right)\right)$$

and consider a clock fixed in frame K' so:

$$x_1'(2) = x_1'(1) \quad -(20)$$

From eqs. (19) and (20):

$$t(2) - t(1) = \gamma\left(t'(2) - t'(1)\right) \quad -(21)$$

i.e.

$$\boxed{\Delta t = \gamma \Delta \tau} \quad -(21)$$

Eq. (21) is the same as eq. (10), obtained from the infinitesimal line element (9)

Eqs. (9), (18) and (21) are obtained self consistently, their origin is:

$$x^\mu x_\mu = x^{\mu\prime} x_\mu' \quad -(22)$$

so

$$c^2 dt^2 - d\underline{r} \cdot d\underline{r} = c^2 dt'^2 - d\underline{r}' \cdot d\underline{r}' \quad -(23)$$

5) In the frame in which the particle is at rest:

$$\underline{dr}' \cdot \underline{dr}' = 0 \quad -(24)$$

because the particle does not move with respect to this frame. So

$$c^2 dt'^2 = c^2 dt^2 - \underline{dr} \cdot \underline{dr} \quad -(25)$$

$$:= c^2 d\tau^2 = ds^2$$

where $d\tau$ is the proper time. The observer frame is the unprimed frame in which

$$\underline{dr} \cdot \underline{dr} = v_N^2 dt^2 \quad -(26)$$

where v_N is the Newtonian velocity. So eq. (22) gives:

$$ds^2 = c^2 d\tau^2 = (c^2 - v_N^2) dt^2 \quad -(27)$$

and

$$d\tau^2 = \left(1 - \frac{v_N^2}{c^2}\right) dt^2 \quad -(28)$$

i.e

$$dt = \gamma \, d\tau \quad -(29)$$

Q.E.D.

Therefore the self consistent definition of γ from eq. (22) is:

$$\boxed{\gamma = \frac{\ell'}{\ell} = \frac{\Delta t}{\Delta \tau}} \quad -(30)$$

A precession can be defined as follows:

$$\boxed{\Delta \phi = 2\pi(\gamma - 1) = 2\pi\left(\frac{\ell'}{\ell} - 1\right) = 2\pi\left(\frac{\Delta t}{\Delta \tau} - 1\right)} \quad -(31)$$

Note carefully that the precession is present from the

) fundamental equation (22). In this case it is defined by the Lorentz factor:

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} = \frac{l'}{l} = \frac{\Delta t}{\Delta \tau} \quad -(32)$$

We name this the precession due to length contraction and time dilation.

Note carefully that there is no need for frame rotation. If frame rotation is considered then γ is changed to:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(33)$$

where

$$v^2 \gg v_N^2 \quad -(34)$$