o(3): Invariance Under Four Rotation and Time Invariance.

· Invariance under four rotation means that:

$$x^\mu x_\mu = x^{\mu\prime} x_\mu^{\prime} \quad - (1)$$

nd for infinitesimals:

$$dx^\mu dx_\mu = dx^{\mu\prime} dx_\mu^{\prime} \quad - (2)$$

o:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} = c^2 dt^{\prime 2} - d\underline{r}^{\prime} \cdot d\underline{r}^{\prime} - (3)$$

ecause the particle does not move with respect to a frame ,xed on the particle, so:

$$d\underline{r}^{\prime} \cdot d\underline{r}^{\prime} = 0. \quad - (4)$$

In Cartesian and plane polar coordinates:

$$d\underline{r} \cdot d\underline{r} = dX^2 + dY^2 = dr^2 + r^2 d\phi^2 - (5)$$

nd the Newtonian velocity is:

$$v_N^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2 - (6)$$

$$= \left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2$$

So:

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\phi^2 = dX^2 + dY^2 = v_N^2 dt^2 - (7)$$

From these equations:

$$ds^2 = (c^2 - v_N^2) dt^2 = c^2 d\tau^2 - (8)$$

Now define

$$c^2 dt_1^2 := (c^2 - v_N^2) dt^2 - (9)$$

It follows that:

$$\boxed{dt_1^2 = d\tau^2.} - (10)$$

Note carefully that this is a re-expression of eq. (1)

Eq. (10) defines the invariance of the infinitesimal time interval squared under four rotation. By definition:

$$x^{\mu}x_{\mu} := c^2 dt_1^2 \quad -(11)$$

$$x^{\mu\prime}x_{\mu}^{\prime} := c^2 d\tau^2 \quad -(12)$$

From eq. (9):

$$dt_1 = \left(1 - \frac{v_N^2}{c^2}\right)^{1/2} dt \quad -(13)$$

So

$$dt = \gamma\, dt_1 = \gamma\, d\tau \quad -(14)$$

The invariance in Eq. (10) can be used to define the precession due to length contraction and time dilation: -(15)

$$\Delta\phi = 2\pi(\gamma - 1) = 2\pi\left(\frac{\ell'}{\ell} - 1\right) = 2\pi\left(\frac{\Delta t}{\Delta\tau} - 1\right).$$

The precession in Eq. (15) can be measured accurately, and is the direct result of the four invariance (2).

Note carefully that de Sitter rotation has not yet been considered. The correct method of developing de Sitter rotation is to consider:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi'^2 \quad -(16)$$

where

$$d\phi' = d\phi + \omega\, dt \quad -(17)$$

and

$$v_\phi = \omega r \quad -(18)$$

From eqs. (16) and (17):

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - 2\omega r^2 d\phi\, dt - \omega^2 r^2 dt^2$$

$$= (c^2 - v_\phi^2)\, dt^2 - 2\omega r^2 d\phi\, dt - dr^2 - r^2 d\phi^2 \quad -(19)$$

By definition:

$$\omega = \frac{d\phi}{dt} \quad -(20)$$

$$d\phi = \omega\, dt \quad -(21)$$

It follows that:

$$2\omega r^2 d\phi\, dt = 2\omega^2 r^2 dt^2 = 2 v_\phi^2 dt^2 \quad -(22)$$

$$ds^2 = (c^2 - 3v_\phi^2)\, dt^2 - (dr^2 + r^2 d\phi^2) \quad -(23)$$

So

$$ds^2 = (c^2 - v_N^2 - 3v_\phi^2)\, dt^2 \quad -(24)$$

The effect of de Sitter rotation can be seen by comparison of eqs. (a) and (24). The effect is to add a term $3v_\phi^2$.

Eq. (24) can be written as:

$$ds^2 = \left(1 - \frac{v^2}{c^2}\right) c^2 dt^2 \quad -(25)$$

where

$$v^2 = v_N^2 + 3v_\phi^2 \quad -(26)$$

Define:

$$dt_2^2 = \left(1 - \frac{v^2}{c^2}\right) dt^2 \quad -(27)$$

and it follows that:

$$\boxed{dt_2^2 = d\tau^2} \quad -(28)$$

The Lorentz factor is replaced by:

$$\gamma_1 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(29)$$

and all precessions are defined by:

$$\boxed{\Delta\phi = 2\pi(\gamma_1 - 1)} \quad -(30)$$

For all precessions, exact agreement with experimental data can be obtained by adjusting

$$v_\phi = \omega r \quad -(31)$$