

407(6): Thomson Precession in the Bohr and Sommerfeld Atom

The Hamiltonian of the Bohr atom is:

$$H = \frac{1}{2} m v^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad (1)$$

$$= \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 \right) - \frac{e^2}{4\pi\epsilon_0 r}$$

The orbits of the Bohr atom are circular, so $\frac{dr}{dt} = 0$ - (2)

and the Hamiltonian is:

$$H = \frac{1}{2} \frac{L^2}{m r^2} - \frac{e^2}{4\pi\epsilon_0 r} \quad (3)$$

and the angular momentum is:

$$L = m r^2 \frac{d\phi}{dt} \quad (4)$$

Bohr introduced the quantization:

$$L = n h \quad (5)$$

where

$$n = 0, 1, 2, 3, \dots \quad (6)$$

H and L are constants of motion so:

$$\frac{dH}{dt} = 0, \quad \frac{dL}{dt} = 0 \quad (7)$$

Using:

$$\frac{dH}{dt} = \frac{dH}{dr} \frac{dr}{dt} \quad (8)$$

it follows from eqs. (2) and (8) that:

$$\frac{dH}{dr} = -\frac{L^2}{m r^3} + \frac{e^2}{4\pi\epsilon_0 r^2} = 0 \quad (9)$$

and this is the Bohr condition:

$$\frac{L^2}{m r^3} = \frac{e^2}{4\pi\epsilon_0 r^2} \quad (10)$$

From eqs. (5) and (10) the Bohr radius is:

$$r_B = \frac{4\pi \epsilon_0 \hbar^2}{m_e e^2} - (11)$$

As in UFT 266 the Bohr velocity is given by:

$$v = v_0 = \omega r = \frac{L}{mr} = \frac{n\hbar}{mr} - (12)$$

Now define the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} - (13)$$

to find that the Bohr radius is:

$$r_B = \frac{n^2 \hbar^2}{m e \alpha} - (14)$$

Therefore the Bohr velocity is:

$$v = c \frac{\alpha}{n} - (15)$$

i.e

$$\boxed{\frac{v}{c} = \frac{\alpha}{n}} - (16)$$

This is Eq. (35) of UFT 266, and was derived for the Schrodinger atom in Note 407(1), Q.E.D.

The Trans half:

$$\frac{\Delta \phi}{2\pi} = \gamma - 1 \xrightarrow{v \ll c} \frac{1}{2} \frac{v^2}{c^2} = \frac{1}{2} \left(\frac{\alpha}{n} \right)^2 - (17)$$

appears to be the Bohr and Schrodinger atoms. The Schrodinger H atom is derived from the same classical Hamiltonian (1) as the Bohr atom, but its quantization is:

$$\underline{p} \phi = -i \hbar \underline{\nabla} \phi - (18)$$

$$3) \hat{H}\psi = \left(-\frac{\hbar^2 \nabla^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r} \right) \psi \quad (19)$$

and $H = \int \psi^* \hat{H} \psi d\tau \quad (20)$

The total energy of the Bohr and Schrodinger atoms is the same. For the Bohr atom, eq. (3) and (10) give:

$$E = \frac{1}{2} \frac{e^2}{4\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r} = -\frac{1}{2} \frac{e^2}{4\pi\epsilon_0 r} \quad (21)$$

also $r = r_B = \frac{n^2 \hbar}{m c \alpha} \quad (22)$

so the energy levels of the Bohr atom are:

$$E = -\frac{1}{2} n c^2 \left(\frac{\alpha}{n} \right)^2 \quad (23)$$

which is also the expectation value of the Bohr velocity is:

$$\langle v \rangle = v = \frac{c \alpha}{n} \quad (24)$$

This is also the expectation value of the Schrodinger velocity as described in Note 407(1).

The Bohr atom contains only one quantum number n , but the Schrodinger atom contains three: n , l , and m_l .

$$m_l = -l, \dots, l \quad (25)$$

The Sommerfeld atom is based on the relativistic

Sommerfeld:

$$H = \gamma m c^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad (26)$$

$$= (\gamma - 1) m c^2 + m c^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

4) So:

$$H_0 = H - mc^2 = (\gamma - 1)mc^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad - (27)$$

i.e

$$\frac{H_0}{mc^2} = (\gamma - 1) - \frac{e^2}{4\pi\epsilon_0 mc^2} \cdot \frac{1}{r}$$
$$= (\gamma - 1) - \left(\frac{d\phi}{h} \right) \frac{1}{r} \quad - (28)$$

where d is the fine structure constant:

$$d = \frac{e^2}{4\pi\epsilon_0 \hbar c} \quad - (29)$$

and

$$\lambda_c = \frac{h}{mc} \quad - (30)$$

is the Compton wavelength.

So

$$\frac{\phi}{h} = \frac{\lambda_c}{2\pi} \quad - (31)$$

and

$$\frac{H_0}{mc^2} = (\gamma - 1) - \frac{d\lambda_c}{2\pi} \cdot \frac{1}{r} \quad - (32)$$

The Thomas precession:

$$\Delta\phi = 2\pi(\gamma - 1) \quad - (33)$$

is the azimuthal precession of the semi-major axis of an elliptical orbit.

In eq. (32):

$$\gamma = \left(1 - \frac{v^2}{c^2} \right)^{-1/2} = \left(1 - \frac{d^2}{n^2} \right)^{-1/2} \quad - (34)$$

Sommerfeld introduced the quantization condition

$$n = n_r + n_\phi \quad - (35)$$

5) where: $n_r = 0, 1, 2, 3, \dots$ } - (36)
 $n_\phi = 1, 2, 3, 4$

so the energy levels are:

$$H_0 = (\gamma - 1)mc^2 - \frac{\hbar c d}{r} \quad - (37)$$

where $(\gamma - 1) = \left(1 - \frac{d^2}{n^2}\right)^{-1/2} - 1$ - (38)
 $= \left(1 - \frac{d^2}{(n_r + n_\phi)^2}\right)^{-1/2} - 1$

The velocity is given by:

$$\frac{v}{c} = \frac{d}{n_r + n_\phi} \quad - (39)$$

and in the low velocity limit:

$$H_0 \rightarrow \frac{1}{2}mv^2 - \frac{\hbar c d}{r} \quad - (40)$$

which gives an elliptical orbital structure:

$$r = \frac{d_0}{1 + \epsilon \cos \phi} \quad - (41)$$

where d_0 is the half right orbit. In analogy with the Newtonian orbital velocity:

$$v^2 = mb \left(\frac{2}{r} - \frac{1}{a} \right) \quad - (42)$$

eq. (40) gives:

$$v^2 = \frac{e}{4\pi\epsilon_0} \left(\frac{2}{r} - \frac{1}{a} \right) \quad - (43)$$

so r can be found in terms of v^2 , and used in eq. (37).