407(6): Thomas Precession in the Bohr and Sommerfeld Atoms

The hamiltonian of the Bohr atom is:

$$H = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad -(1)$$

$$= \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) - \frac{e^2}{4\pi\epsilon_0 r}$$

The orbitals of the Bohr atom are circular, so

$$\frac{dr}{dt} = 0 \quad -(2)$$

and the hamiltonian is:

$$H = \frac{1}{2}\frac{L^2}{mr^2} - \frac{e^2}{4\pi\epsilon_0 r} \quad -(3)$$

where the angular momentum is:

$$L = mr^2\frac{d\phi}{dt} \quad -(4)$$

Bohr introduced the quantization:

$$L = n\hbar \quad -(5)$$

where

$$n = 0, 1, 2, 3, \ldots \quad -(6)$$

H and L are constants of motion so:

$$\frac{dH}{dt} = 0, \quad \frac{dL}{dt} = 0 \quad -(7)$$

Using:

$$\frac{dH}{dt} = \frac{dH}{dr}\frac{dr}{dt} \quad -(8)$$

it follows from eqs. (2) and (8) that:

$$\frac{dH}{dr} = -\frac{L^2}{mr^3} + \frac{e^2}{4\pi\epsilon_0 r^2} = 0 \quad -(9)$$

and this is the Bohr condition:

$$\frac{L^2}{mr^3} = \frac{e^2}{4\pi\epsilon_0 r^2} \quad -(10)$$

From eqs. (5) and (10) the Bohr radius is:

2)

$$r_B = \frac{4\pi\epsilon_0 \hbar^2}{m e^2} \quad -(11)$$

As in 4FT 266 the Bohr velocity is given by:

$$v = v_\theta = \omega r = \frac{L}{mr} = \frac{n\hbar}{mr} \quad -(12)$$

Now define the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\hbar c\epsilon_0} \quad -(13)$$

to find that the Bohr radius is:

$$r_B = \frac{n^2\hbar}{mc\alpha} \quad -(14)$$

Therefore the Bohr velocity is:

$$v = \frac{c\alpha}{n} \quad -(15)$$

i.e

$$\boxed{\frac{v}{c} = \frac{\alpha}{n}} \quad -(16)$$

This is Eq. (35) of 4FT 266, and was derived for the Schroedinger atom in Note 407(1), Q.E.D.

The Thomas half:

$$\frac{\Delta\phi}{2\pi} = \gamma - 1 \xrightarrow[v \ll c]{} \frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2}\left(\frac{\alpha}{n}\right)^2 \quad -(17)$$

appears both in the Bohr and Schroedinger atoms.

The Schroedinger H atom is derived from the same classical hamiltonian (1) as the Bohr atom, but in the former the quantization is:

$$\underline{p}\,\psi = -i\hbar\underline{\nabla}\psi \quad -(18)$$

So

3)
$$\hat{H}\phi = \left(-\frac{\hbar^2 \nabla^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r} \right) \phi \quad -(19)$$

and
$$H = \int \phi^* \hat{H} \phi \, d\tau \quad -(20)$$

The total energy of the Bohr and Schroedinger atoms is the same. For the Bohr atom, eq. (3) and (10) give:
$$E = \frac{1}{2} \frac{e^2}{4\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r} = -\frac{1}{2} \frac{e^2}{4\pi\epsilon_0 r} \quad -(21)$$

where
$$r = r_B = \frac{n^2 \hbar}{mc\alpha} \quad -(22)$$

so the energy levels of the Bohr atom are:
$$E = -\frac{1}{2} mc^2 \left(\frac{\alpha}{n} \right)^2 \quad -(23)$$
$$= -\frac{1}{2} m \langle v^2 \rangle$$

where the expectation value of the Bohr velocity is:
$$\langle v \rangle = v = \frac{c\alpha}{n} \quad -(24)$$

This is also the expectation value of the Schroedinger velocity as described in Note 407(1).

The Bohr atom contains only one quantum number n, but the Schroedinger atom contains three: n, l, and
$$m_l = -l, \ldots l \quad -(25)$$

The Sommerfeld atom is based on the relativistic hamiltonian:
$$H = \gamma mc^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad -(26)$$
$$= (\gamma - 1) mc^2 + mc^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

4) So:

$$H_0 = H - mc^2 = (\gamma - 1) mc^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad -(27)$$

i.e

$$\frac{H_0}{mc^2} = (\gamma - 1) - \frac{e^2}{4\pi\epsilon_0 mc^2} \cdot \frac{1}{r}$$

$$= (\gamma - 1) - \left(\alpha \frac{\hbar}{mc}\right) \frac{1}{r} \quad -(28)$$

where α is the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\hbar c\epsilon_0} \quad -(29)$$

and

$$\lambda_c = \frac{h}{mc} \quad -(30)$$

is the Compton wavelength.

So

$$\frac{\hbar}{mc} = \frac{\lambda_c}{2\pi} \quad -(31)$$

and

$$\frac{H_0}{mc^2} = (\gamma - 1) - \frac{\alpha \lambda_c}{2\pi} \cdot \frac{1}{r} \quad -(32)$$

The Thomas precession:

$$\Delta\phi = 2\pi(\gamma - 1) \quad -(33)$$

is the azimuthal precession of the semi major axis of an elliptical orbital.

In eq. (32):

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \left(1 - \frac{\alpha^2}{n^2}\right)^{-1/2} \quad -(34)$$

Sommerfeld introduced the quantization condition

$$n = n_r + n_\phi \quad -(35)$$

5) where:

$$\left.\begin{aligned} n_r &= 0, 1, 2, 3 \ldots \\ n_\phi &= 1, 2, 3, 4 \end{aligned}\right\} \quad -(36)$$

so the energy levels are:

$$H_0 = (\gamma - 1) mc^2 - \frac{\hbar c \alpha}{r} \quad -(37)$$

where

$$\begin{aligned} (\gamma - 1) &= \left(1 - \frac{\alpha^2}{n^2}\right)^{-1/2} - 1 \\ &= \left(1 - \frac{\alpha^2}{(n_r + n_\phi)^2}\right)^{-1/2} - 1 \end{aligned} \quad -(38)$$

The velocity is given by:

$$\frac{v}{c} = \frac{\alpha}{n_r + n_\phi} \quad -(39)$$

and in the low velocity limit:

$$H_0 \rightarrow \frac{1}{2} mv^2 - \frac{\hbar c \alpha}{r} \quad -(40)$$

which gives an elliptical orbital structure:

$$r = \frac{\alpha_0}{1 + \epsilon \cos \phi} \quad -(41)$$

where α_0 is the half right latitude. In analogy with the Newtonian orbital velocity:

$$v^2 = mG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(42)$$

eq. (40) gives:

$$v^2 = \frac{e}{4\pi\epsilon_0}\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(43)$$

so r can be found in terms of v^2, and used in eq. (37).