405(5): Solution of Differential Equation for $\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$

This is eq. (1) of Note 405(4):

$$\Delta \phi = \frac{r^2}{2}\left(\frac{\omega}{r} - \frac{\partial \omega}{\partial r}\right) \quad -(1)$$

$$= \frac{4}{3}\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} - \frac{1}{3r}\frac{\partial}{\partial r}\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$$

If

$$f(r) := \langle \delta \underline{r} \cdot \delta \underline{r} \rangle \quad -(2)$$

Eq. (1) can be written as:

$$4 f(r) - r \frac{\partial f(r)}{\partial r} = 3 r^2 \Delta \phi. \quad -(3)$$

According to the Wolfram site a differential equations the solution of Eq. (3) is:

$$\langle \delta \underline{r} \cdot \delta \underline{r} \rangle = \frac{3}{2} r^2 \Delta \phi + C_1 r^4 \quad -(4)$$

where C_1 is a constant of integration. If C_1 is chosen to be zero then:

$$\boxed{\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} = \frac{3}{2}\Delta \phi} \quad -(5)$$

which shows that any precession is due to the isotropically averaged vacuum fluctuation $\langle \delta \underline{r} \cdot \delta \underline{r} \rangle$.

The starting equation is:

$$\underline{F} = -m \underline{\nabla} \phi + m \underline{\omega} \phi \quad -(6)$$

and this is solved in the apsidal approximation

2) for a nearly circular orbit.

If each orbit is considered to be generated by a Thomas precession of velocity:

$$v = \omega_0 r \quad -(7)$$

then

$$\Delta\phi \sim \pi\left(\frac{v}{c}\right)^2 \quad -(8)$$

From eqs. (5) and (8):

$$\boxed{\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2} = \frac{3}{2}\pi\left(\frac{v}{c}\right)^2} \quad -(9)$$

<u>Application to Gravity Probe B.</u>

From eq. (21) of the previous note the velocities of the gravitational, de Sitter and Lense Thirring precessions are:

$$\left.\begin{aligned} v\,(\text{grav}) &= 1.84 \times 10^5 \text{ ms}^{-1} \\ v\,(\text{de Sitter}) &= 1.25 \times 10^5 \text{ ms}^{-1} \\ v\,(\text{Lense Thirring}) &= 5.70 \times 10^3 \text{ ms}^{-1} \end{aligned}\right] \quad -(10)$$

So:

$$\left(\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2}\right)_{\text{grav.}} = 1.78 \times 10^{-6} \quad -(11)$$

$$\left(\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2}\right)_{\text{de Sitter}} = 8.19 \times 10^{-7} \quad -(12)$$

$$\left(\frac{\langle \delta \underline{r} \cdot \delta \underline{r} \rangle}{r^2}\right)_{\text{Lense Thirring}} = 1.70 \times 10^{-9} \quad -(13)$$