81(5): Solution for the Static Gravitational Field

In this case the complete set of equations is:

$$\underline{g} = -\underline{\nabla}\,\Phi + \underline{\omega}\,\Phi = -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \quad -(1)$$

where the gravitational four potential is:

$$Q^\mu = \left(\frac{\Phi}{c}, \underline{Q} \right) \quad -(2)$$

and where the spin connection four-vector is:

$$\omega^\mu = \left(\frac{\omega_0}{c}, \underline{\omega} \right) \quad -(3)$$

S.I. Units

$$\underline{g} = m\,s^{-2}, \quad \Phi = m^2 s^{-2}, \quad -(4)$$

$$\underline{Q} = m\,s^{-1}, \quad \omega_0 = s^{-1}$$

Similarly, for electrodynamics:

$$\underline{E} = -\underline{\nabla}\,\phi + \underline{\omega}\,\phi = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A} \quad -(5)$$

where

$$A^\mu = \left(\frac{\phi}{c}, \underline{A} \right) \quad -(6)$$

and where ω^μ is the same (defined in Eq. (3)). The e/m units are:

$$\underline{E} = \text{volt}\,m^{-1} = J\,C^{-1} m^{-1} \quad -(7)$$

$$\underline{A} = J\,s\,C^{-1} m^{-1} \quad -(8)$$

$$\phi = \text{volt} = J\,C^{-1} \quad -(9)$$

The units of the spin connection are:

$$\omega_0 = s^{-1}, \quad \underline{\omega} = m^{-1} \quad -(10)$$

both for gravitation and electromagnetism.

The gravitostatic field equations are:

$$\underline{\nabla} \times \underline{g} = \underline{0} \quad -(11)$$

$$\underline{\nabla} \cdot \underline{g} = 4\pi G \rho_m \quad -(12)$$

$$\frac{\partial \underline{g}}{\partial t} = \underline{0} \quad -(13)$$

and

$$\underline{\Omega} = \underline{\nabla} \times \underline{Q} - \underline{\omega} \times \underline{Q} = \underline{0} \quad -(14)$$

where $\underline{\Omega}$ is the gravitomagnetic field. Therefore:

$$\underline{\nabla} \times \underline{Q} = \underline{\omega} \times \underline{Q} \quad -(15)$$

From eqs. (1) and (11):

$$\frac{\partial}{\partial t} \underline{\nabla} \times \underline{Q} + \underline{\nabla} \times (\omega_0 \underline{Q}) = \underline{0} \quad -(16)$$

Assume that $\underline{g}$ is time independent. It follows that:

$$\frac{\partial \underline{Q}}{\partial t} = \underline{0} \quad -(17)$$

and from eq. (16):

$$\underline{\nabla} \times (\omega_0 \underline{Q}) = \omega_0 \underline{\nabla} \times \underline{Q} + \underline{Q} \times \underline{\nabla} \omega_0 = \underline{0} \quad -(18)$$

Now write out the Cartesian components of Eqs. (15), (18) and (12) to obtain the following seven simultaneous equations in seven unknowns:

$$\frac{\partial Q_Z}{\partial Y} - \frac{\partial Q_Y}{\partial Z} = \omega_Y Q_Z - \omega_Z Q_Y \quad -(19)$$

$$\frac{\partial Q_X}{\partial Z} - \frac{\partial Q_Z}{\partial X} = \omega_Z Q_X - \omega_X Q_Z \quad -(20)$$

$$\frac{\partial Q_Y}{\partial X} - \frac{\partial Q_X}{\partial Y} = \omega_X Q_Y - \omega_Y Q_X \quad -(21)$$

$$\omega_0 \left(\frac{\partial q_Z}{\partial Y} - \frac{\partial q_Y}{\partial Z} \right) + q_Z \frac{\partial \omega_0}{\partial Y} - q_Y \frac{\partial \omega_0}{\partial Z} = 0 \quad -(22)$$

$$\omega_0 \left(\frac{\partial q_X}{\partial Z} - \frac{\partial q_Z}{\partial X} \right) + q_X \frac{\partial \omega_0}{\partial Z} - q_Z \frac{\partial \omega_0}{\partial X} = 0 \quad -(23)$$

$$\omega_0 \left(\frac{\partial q_Y}{\partial Z} - \frac{\partial q_X}{\partial Y} \right) + q_Y \frac{\partial \omega_0}{\partial X} - q_X \frac{\partial \omega_0}{\partial Y} = 0 \quad -(24)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial q_X}{\partial X} + \frac{\partial q_Y}{\partial Y} + \frac{\partial q_Z}{\partial Z} \right) + \omega_0 \left(\frac{\partial q_X}{\partial X} + \frac{\partial q_Y}{\partial Y} + \frac{\partial q_Z}{\partial Z} \right) + q_X \frac{\partial \omega_0}{\partial X} + q_Y \frac{\partial \omega_0}{\partial Y} + q_Z \frac{\partial \omega_0}{\partial Z} = -4\pi G \rho_m \quad -(25)$$

The seven unknowns are q_X, q_Y, q_Z, ω_0, ω_X, ω_Y and ω_Z. It is assumed that ρ_m is known experimentally. These equations are ECE2 covariant and are equations of general relativity. The rigorous solution addresses any problem in gravitation.

Example Solution

Assume that:

$$\underline{g} = -\underline{\nabla}\,\Phi + \underline{\omega}\,\Phi = -MG\frac{\underline{r}}{r^3} \quad -(26)$$

This equation has a solution:

$$\Phi = -\frac{MG}{2r} \quad -(27)$$

$$\underline{\omega} = \frac{\underline{r}}{r^2} \quad -(28)$$

so

$$-\underline{\nabla}\,\Phi = -\frac{MG}{2}\frac{\underline{r}}{r^3} \quad -(29)$$

and
$$\underline{\omega}\,\underline{\Phi} = -\frac{mG}{2}\frac{\underline{r}}{r^3} \quad -(30)$$

It follows that the gravitational force is:

$$\underline{F} = m\underline{g} = -mMG\frac{\underline{r}}{r^3} \quad -(31)$$
$$= -m\omega_0\underline{Q}$$

From eq. (13):

$$\frac{\partial \omega_0}{\partial t} = 0 \quad -(32)$$

From eq. (11):

$$\underline{\nabla} \times (\omega_0\underline{Q}) = \omega_0\underline{\nabla}\times\underline{Q} + \underline{Q}\times\underline{\nabla}\omega_0 = \underline{0} \quad -(33)$$

which follows from eq. (16) with the assumption:

$$\frac{\partial \underline{Q}}{\partial t} = \underline{0}. \quad -(34)$$

Using:
$$\underline{Q} = -\frac{\underline{g}}{\omega_0} \quad -(35)$$
$$= \frac{MG}{\omega_0}\frac{\underline{r}}{r^3}$$

it follows from eq. (11) that:

$$\omega_0\underline{\nabla}\times\underline{Q} = \underline{0} \quad -(36)$$

so from eq. (16):

$$\underline{Q}\times\underline{\nabla}\omega_0 = \underline{0} \quad -(37)$$

A possible solution of eq. (37) is:

$$\underline{Q} = \frac{MG}{\omega_0 r^3} \underline{r} \quad -(38)$$

where:

$$\frac{\underline{r}}{r^3} = \frac{1}{c} \underline{\nabla} \omega_0 \quad -(39)$$

so

$$\omega_0 = -\frac{c}{r} \quad -(40)$$

The spin connection four vector is therefore:

$$\omega^\mu = \left(-\frac{1}{r}, \frac{\underline{r}}{r^2} \right) \quad -(41)$$

and eq. (31) gives an elliptical orbit if the spin connection ω^μ is defined by eq. (41):

$$\underline{F} = m\underline{g} = -mMG\frac{\underline{r}}{r^3} = -m\omega_0 \underline{Q} \quad -(42)$$

The elliptical orbit is obtained by assuming that:

$$\underline{F} = m\left(-\underline{\nabla}\Phi + \underline{\omega}\Phi \right) = -mMG\frac{\underline{r}}{r^3} = -m\omega_0\underline{Q} \quad -(43)$$

and

$$\frac{\partial \underline{Q}}{\partial t} = \underline{0} \quad -(44)$$

A precessing elliptical orbit is obtained by a different choice of $\omega_0 \underline{Q}$, and this will be discussed in the next note.