1(4): Solution for Static Electric Field

In this case the complete set of equations is:

$$\underline{E} = -\underline{\nabla}\phi + \underline{\omega}\phi = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A}, \quad -(1)$$

y antisymmetry, and $\quad A^\mu = \left(\frac{\phi}{c}, \underline{A}\right); \; \omega^\mu = \left(\frac{\omega_0}{c}, \underline{\omega}\right) \quad -(1a)$

$$\underline{\nabla} \times \underline{E} = 0 \quad -(2)$$

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad -(3)$$

$$\frac{\partial \underline{E}}{\partial t} = \underline{0} \quad -(4)$$

and

$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} = \underline{0}. \quad -(5)$$

There are seven equations in seven unknowns as follows:

m eq. (5):

$$\underline{\nabla} \times \underline{A} = \underline{\omega} \times \underline{A} \quad -(6)$$

i.e

$$\frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z} = \omega_Y A_Z - \omega_Z A_Y \quad -(7)$$

$$\frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X} = \omega_Z A_X - \omega_X A_Z \quad -(8)$$

$$\frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y} = \omega_X A_Y - \omega_Y A_X \quad -(9)$$

Fm eqs. (1) and (2):

$$\frac{\partial}{\partial t} \underline{\nabla} \times \underline{A} + \underline{\nabla} \times (\omega_0 \underline{A}) = \underline{0} \quad -(10)$$

Assume that $\underline{A}$ has no time dependence:

$$\frac{\partial \underline{A}}{\partial t} = \underline{0} \quad -(11)$$

because the static electric field is time independent by definition. It follows that:

$$\nabla \times (\omega_0 \underline{A}) = \omega_0 \underline{\nabla} \times \underline{A} + \underline{A} \times \underline{\nabla} \omega_0 = \underline{0} \quad -(12)$$

i.e.

$$\omega_0 \left(\frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z} \right) + A_Z \frac{\partial \omega_0}{\partial Y} - A_Y \frac{\partial \omega_0}{\partial Z} = 0 \quad -(13)$$

$$\omega_0 \left(\frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X} \right) + A_X \frac{\partial \omega_0}{\partial Z} - A_Z \frac{\partial \omega_0}{\partial X} = 0 \quad -(14)$$

$$\omega_0 \left(\frac{\partial A_Y}{\partial Z} - \frac{\partial A_X}{\partial Y} \right) + A_Y \frac{\partial \omega_0}{\partial X} - A_X \frac{\partial \omega_0}{\partial Y} = 0 \quad -(15)$$

Finally use the Coulomb law:

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad -(16)$$

i.e.

$$\frac{\partial}{\partial t} \underline{\nabla} \cdot \underline{A} + \omega_0 \underline{\nabla} \cdot \underline{A} + \underline{A} \cdot \underline{\nabla} \omega_0 = -\rho / \epsilon_0 \quad -(17)$$

which is:

$$\frac{\partial}{\partial t} \left(\frac{\partial A_X}{\partial X} + \frac{\partial A_Y}{\partial Y} + \frac{\partial A_Z}{\partial Z} \right) + \omega_0 \left(\frac{\partial A_X}{\partial X} + \frac{\partial A_Y}{\partial Y} + \frac{\partial A_Z}{\partial Z} \right) + A_X \frac{\partial \omega_0}{\partial X} + A_Y \frac{\partial \omega_0}{\partial Y} + A_Z \frac{\partial \omega_0}{\partial Z}$$

$$= -\frac{\rho}{\epsilon_0} \quad -(18)$$

Eqs. (7) to (9), (13) to (15) and (17) are seven equations in the seven unknowns A_X, A_Y, A_Z, ω_0, ω_X, ω_Y and ω_Z, given that ρ / ϵ_0 is known experimentally. So any situation in physics and engineering can be addressed.

Example Solution

Assume the Coulomb field, which is known experimentally with great accuracy. Then:

$$\underline{E} = -\underline{\nabla}\phi + \underline{\omega}\phi = -\frac{e_1 e_2}{4\pi\epsilon_0 r^3}\underline{r} \quad -(19)$$

This has the solution:

$$\phi = -\frac{e_1 e_2}{8\pi\epsilon_0 r} \quad -(20)$$

and

$$\underline{\omega} = \frac{\underline{r}}{r^2} \quad -(21)$$

so:

$$\underline{\nabla}\phi = -\frac{e_1 e_2}{8\pi\epsilon_0 r^3}\underline{r} \quad -(22)$$

and

$$\underline{\omega}\phi = -\frac{e_1 e_2}{8\pi\epsilon_0 r^3}\underline{r} \quad -(23)$$

and:

$$\underline{E} = -\frac{e_1 e_2}{4\pi\epsilon_0 r^3}\underline{r} = -\omega_0 \underline{A} \quad -(24)$$

From eq. (4):

$$\underline{A}\frac{\partial \omega_0}{\partial t} = \underline{0} \quad -(25)$$

so

$$\frac{\partial \omega_0}{\partial t} = 0 \quad -(26)$$

From eq. (2):

$$\underline{\nabla}\times(\omega_0 \underline{A}) = \omega_0 \underline{\nabla}\times\underline{A} + \underline{A}\times\underline{\nabla}\omega_0 = \underline{0} \quad -(27)$$

4) Using:

$$\underline{A} = \frac{e_1 e_2}{4\pi\epsilon_0 \omega_0} \frac{\underline{r}}{r^3} \quad -(28)$$

it follows that:

$$\omega_0 \underline{\nabla} \times \underline{A} = \underline{0} \quad -(29)$$

so:

$$\underline{A} \times \underline{\nabla} \omega_0 = \underline{0}. \quad -(30)$$

A possible solution is:

$$\underline{A} = \frac{e_1 e_2}{4\pi\epsilon_0 \omega_0} \frac{\underline{\nabla}\omega_0}{c} \quad -(31)$$

so

$$\underline{\nabla}\omega_0 = c\frac{\underline{r}}{r^3} \quad -(32)$$

and

$$\omega_0 = -\frac{c}{r} \quad -(33)$$

<u>The Complete Solution</u>

$$\underline{E} = -\omega_0 \underline{A} = -\frac{e_1 e_2}{4\pi\epsilon_0 r^3}\underline{r} \quad -(34)$$

$$\omega_0 = -\frac{c}{r}, \quad \underline{\omega} = \frac{\underline{r}}{r^3} \quad -(35)$$

$$\underline{A} = \frac{e_1 e_2}{4\pi\epsilon_0 r^2}\underline{r} \quad -(36)$$

$$\phi = -\frac{e_1 e_2}{8\pi\epsilon_0 r} \quad -(37)$$

$$\partial \underline{A} / \partial t = \underline{0}$$