381(3): Application to a Static Magnetic Field

As discussed in Note 381(2), there are particular solutions of the general antisymmetry law of ECE2:

$$\left(\frac{\partial}{\partial Y} - \omega_Y\right) A_Z = -\left(\frac{\partial}{\partial Z} - \omega_Z\right) A_Y \quad -(1)$$

$$\left(\frac{\partial}{\partial Z} - \omega_Z\right) A_X = -\left(\frac{\partial}{\partial X} - \omega_X\right) A_Z \quad -(2)$$

$$\left(\frac{\partial}{\partial X} - \omega_X\right) A_Y = -\left(\frac{\partial}{\partial Y} - \omega_Y\right) A_X \quad -(3)$$

In two dimensions X and Y:

$$\left(\frac{\partial}{\partial X} - \omega_X\right) A_Y = -\left(\frac{\partial}{\partial Y} - \omega_Y\right) A_X. \quad -(4)$$

This equation has the particular solutions:

$$\frac{\partial A_Y}{\partial X} = -\frac{\partial A_X}{\partial Y} \quad -(5)$$

and

$$-\omega_X A_Y = \omega_Y A_X. \quad -(6)$$

It also has the particular solutions:

$$\frac{\partial A_Y}{\partial X} = \omega_Y A_X \quad -(7)$$

and

$$\frac{\partial A_X}{\partial Y} = \omega_X A_Y. \quad -(8)$$

Eqs. (5) to (8) are self consistent.

Consider the planar potential:

$$\underline{A} = \frac{B^{(0)}}{2}\left(-Y\underline{i} + X\underline{j}\right) \quad -(9)$$

It follows that:

$$\frac{\partial A_Y}{\partial X} = -\frac{\partial A_X}{\partial Y}, \quad -(10)$$

which is Eq. (5), Q.E.D. The antisymmetry law's particular solution is obeyed by eq. (9).

We have:

$$A_X = -Y, \; A_Y = X \quad -(11)$$

so eq. (6) gives:

$$Y \omega_X = X \omega_Y \quad -(12)$$

Furthermore:

$$\frac{\partial A_X}{\partial Y} = -1, \; \frac{\partial A_Y}{\partial X} = 1 \quad -(13)$$

From eqs. (7), (8), (11) and (13):

$$\boxed{\omega_Y = -\frac{1}{Y}, \; \omega_X = -\frac{1}{X}} \quad -(14)$$

and the spin connection vector is:

$$\underline{\omega} = -\left(\frac{1}{X}\underline{i} + \frac{1}{Y}\underline{j}\right) \quad -(15)$$

Therefore:

$$\underline{\omega} \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_X & \omega_Y & 0 \\ A_X & A_Y & 0 \end{vmatrix}$$

$$= \left(\omega_X A_Y - \omega_Y A_X\right)\underline{k}$$

$$= \frac{B^{(0)}}{2}\left(-1-1\right)\underline{k} \quad -(16)$$

$$= -B^{(0)}\underline{k}$$

3) The magnetic field is therefore:

$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} \qquad -(17)$$
$$= 2B^{(0)} \underline{k}$$

which is a static magnetic flux density in the $\underline{k}$ axis, Q.E.D.

The electric field strength is zero:

$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A} = \underline{0}. \qquad -(18)$$

Using:

$$\frac{\partial \underline{A}}{\partial t} = \underline{0} \qquad -(19)$$

it follows that:

$$\boxed{\omega_0 = 0} \qquad -(20)$$

Therefore the equation:

$$\frac{\partial}{\partial t}(\underline{\omega} \times \underline{A}) + \underline{\nabla} + (\omega_0 \underline{A}) = \underline{0} \qquad -(21)$$

is true, because:

$$\frac{\partial}{\partial t}(\underline{\omega} \times \underline{A}) = \underline{0} \qquad -(22)$$

Therefore the equations:

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad -(23)$$
$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \qquad -(24)$$
$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} \qquad -(25)$$
$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A} \qquad -(26)$$

4) are all obeyed, Q.E.D.

The Ampere law is:

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \quad -(27)$$

and for:

$$\underline{B} = 2B^{(0)} \underline{k} \quad -(28)$$

the net current density is zero, which is self consistent with the fact that there is no charge density ρ because $\underline{E}$ is zero.

Complete Solution

$$\underline{B} = 2B^{(0)} \underline{k}, \quad \underline{E} = \underline{0}, \quad -(29)$$

$$\underline{A} = \frac{B^{(0)}}{2} \left(-Y \underline{i} + X \underline{j} \right) \quad -(30)$$

$$\omega_X = -\frac{1}{Y}, \quad \omega_Y = -\frac{1}{X} \quad -(31)$$

In the standard model of electrodynamics, if $\underline{A}$ is assumed then:

$$\underline{B}_{sm} = B^{(0)} \underline{k}, \quad \underline{E}_{sm} = \underline{0}, \quad -(32)$$

$$\omega_{Xsm} = \omega_{Ysm} = 0 \quad -(33)$$