Note 381(1): Complete Solution of the ECE2 Field Equations

For electrodynamics the complete field equations are:

$$\underline{\nabla} \cdot \underline{B} = \underline{\kappa} \cdot \underline{B} \quad -(1)$$

$$\underline{\nabla} \cdot \underline{E} = \underline{\kappa} \cdot \underline{E} = \rho / \epsilon_0 \quad -(2)$$

$$\frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times \underline{E} = -\left( \kappa_0 c \underline{B} + \underline{\kappa} \times \underline{E} \right) \quad -(3)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \frac{\kappa_0}{c} \underline{E} + \underline{\kappa} \times \underline{B} \quad -(4)$$

$$= \mu_0 \underline{J}$$

where

$$\kappa_0 = 2 \left( \frac{q_0}{r^{(0)}} - \omega_0 \right) = 2 \left( \frac{A_0}{A^{(0)} r^{(0)}} - \omega_0 \right) \quad -(5)$$

and

$$\underline{\kappa} = 2 \left( \frac{\underline{q}}{r^{(0)}} - \underline{\omega} \right) = 2 \left( \frac{\underline{A}}{A^{(0)} r^{(0)}} - \underline{\omega} \right) \quad -(6)$$

in the notation of UFT 316 and UFT 317.

From the ECE Hypothesis:

$$\underline{A} = A^{(0)} \underline{q} \quad -(7)$$

$$A_0 = A^{(0)} q_0 \quad -(8)$$

These equations allow for the existence of a magnetic charge current density in general. If it is assumed that this quantity is zero then in general:

$$\underline{\kappa} \cdot \underline{B} = 0 \quad -(9)$$

$$\kappa_0 c \underline{B} + \underline{\kappa} \times \underline{E} = \underline{0} \quad -(10)$$

In free space:

2)

$$\underline{\kappa} \cdot \underline{E} = \rho / \epsilon_0 = 0 \quad -(11)$$

and

$$\frac{\kappa_0}{c} \underline{E} + \underline{\kappa} \times \underline{B} = \mu_0 \underline{J} = \underline{0} \quad -(12)$$

So in free space:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(13)$$

$$\underline{\nabla} \cdot \underline{E} = 0 \quad -(14)$$

$$\frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times \underline{E} = \underline{0} \quad -(15)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \underline{0} \quad -(16)$$

and

$$\underline{\kappa} \cdot \underline{B} = 0 \quad -(17)$$

$$\underline{\kappa} \cdot \underline{E} = 0 \quad -(18)$$

$$\kappa_0 c \underline{B} + \underline{\kappa} \times \underline{E} = \underline{0} \quad -(19)$$

$$\frac{\kappa_0}{c} \underline{E} + \underline{\kappa} \times \underline{B} = \underline{0} \quad -(20)$$

Eqs (13) to (16) are satisfied by the plane waves:

$$\underline{E} = \frac{E^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \quad -(21)$$

$$\underline{B} = \frac{B^{(0)}}{\sqrt{2}} (i\underline{i} + \underline{j}) e^{i\phi} \quad -(22)$$

where

$$\phi = \omega t - \kappa_Z Z \quad -(23)$$

Eqs. (17) to (20) are satisfied by:

$$\underline{\kappa} = \underline{0} \quad -(24)$$

$$\kappa_0 = 0 \quad -(25)$$

So
$$q_0 = r^{(0)}\omega_0 \quad -(26)$$
and
$$\underline{q} = r^{(0)}\underline{\omega} \quad -(27)$$
The electric field strength is given by:
$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A} \quad -(28)$$
and the magnetic flux density by:
$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} \quad -(29)$$
When used with:
$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(30)$$
eqs. (28) and (29) imply:
$$\frac{\partial}{\partial t}\left(\underline{\omega} \times \underline{A}\right) + \underline{\nabla} \times \left(\omega_0 \underline{A}\right) = \underline{0} \quad -(31)$$
The antisymmetry laws from eq. (29) imply:
$$\left(\frac{\partial}{\partial Y} - \omega_Y\right) A_Z = -\left(\frac{\partial}{\partial Z} - \omega_Z\right) A_Y \quad -(32)$$
$$\left(\frac{\partial}{\partial Z} - \omega_Z\right) A_X = -\left(\frac{\partial}{\partial X} - \omega_X\right) A_Z \quad -(33)$$
$$\left(\frac{\partial}{\partial X} - \omega_X\right) A_Y = -\left(\frac{\partial}{\partial Y} - \omega_Y\right) A_X \quad -(34)$$
If:
$$\underline{A} = \frac{A^{(0)}}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) e^{i\phi} \quad -(35)$$
The antisymmetry laws reduce to:
$$\frac{\partial A_X}{\partial Z} = \omega_Z A_X \quad -(36)$$
$$= -i\kappa_Z A_X$$

$$\frac{\partial A_Y}{\partial Z} = \omega_Z A_Y = -i \kappa_Z A_Y \quad - (37)$$

and

$$-\omega_X A_Y = \omega_Y A_X \quad - (38)$$

It follows as in UFT380 that:

$$\omega_Z = \cancel{\phi}\, -i\kappa_Z \quad - (39) \qquad \text{Real } \omega_Z = 0 \quad - (39a)$$

and

$$\underline{\omega} = \frac{\omega^{(0)}}{\sqrt{2}} \left( -i\,\underline{i} + \underline{j} \right) e^{i\phi} \quad - (40)$$

From eq. (37):

$$\underline{\omega} = \frac{\underline{A}^*}{A^{(0)} r^{(0)}} \quad - (41)$$

where $\underline{A}^*$ is the complex conjugate of $\underline{A}$:

$$\underline{A}^* = \frac{A^{(0)}}{\sqrt{2}} \left( -i\,\underline{i} + \underline{j} \right) e^{-i\phi} \quad - (42)$$

Finally, Eq. (31) implies that:

$$\omega_0 = 0 \quad - (43)$$

__The Complete Solution__

$$\underline{E} = \frac{E^{(0)}}{\sqrt{2}} \left( \underline{i} - i\,\underline{j} \right) e^{i\phi} \quad - (44)$$

$$\underline{B} = \frac{B^{(0)}}{\sqrt{2}} \left( i\,\underline{i} + \underline{j} \right) e^{i\phi} \quad - (45)$$

$$\underline{A} = \frac{A^{(0)}}{\sqrt{2}} \left( i\,\underline{i} + \underline{j} \right) e^{i\phi} \quad - (46)$$

$$\kappa_0 = 0 \quad - (47)$$

$$\underline{\kappa} = \underline{0} \quad - (48)$$

5)

$$\omega_0 = 0 \quad -(49)$$

$$\underline{\omega} = \frac{\omega^{(0)}}{\sqrt{2}}\left(-i\underline{i} + \underline{j}\right)e^{-i\phi} \quad -(50)$$

Therefore in the absence of magnetic charge current density, the electromagnetic plane wave in free space is accompanied by a spin connection plane wave (50).

In general, the condition for the absence of magnetic charge current density is given by eqs. (9) and (10), and in general, $K^\mu = (K_0, \underline{K})$ is not zero.

The above example can be checked by computer algebra. Note carefully that in Eq. (31)

$$\underline{\omega} \times \underline{A} = \frac{\omega^{(0)}A^{(0)}}{2}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & i & 0 \\ 1 & -i & 0 \end{vmatrix} \quad -(51)$$

$$= -i\omega^{(0)}A^{(0)}$$

Its real part is zero, and

$$\frac{\partial}{\partial t}\left(\underline{\omega} \times \underline{A}\right) = \underline{0} \quad -(52)$$

so eq. (31) is satisfied, using eq. (49).

As shown in UFT380, the homogeneous field equations:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(53)$$

and

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(54)$$

together with eq. (8) and eqs. (5), (6) and (7)

) give seven equations in seven unknowns, the three components of $\underline{A}$ and the four components of the spin connection vector. This is a completely general method.

Having found $\underline{A}$ and $\omega^\mu$, the electric and magnetic fields $\underline{E}$ and $\underline{B}$ can be found. The $\kappa^\mu$ vector can be found from eqs. (5) and (6). In the absence of a magnetic charge / current density, $\kappa^\mu = (\kappa_0, \underline{\kappa})$, $\underline{E}$ and $\underline{B}$ must obey eqs. (9) and (10). So this constraint defines the distance parameter $r^{(0)}$ in general.

So any situation in the physical sciences and engineering can be described as above.

The inhomogeneous field equations are:

$$\underline{\nabla} \cdot \underline{E} = \underline{\kappa} \cdot \underline{E} = \rho / \epsilon_0 \quad -(55)$$

and

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \frac{\kappa_0}{c} \underline{E} + \underline{\kappa} \times \underline{B} = \mu_0 \underline{J} \quad -(56)$$

Since $\kappa_0$ and $\underline{\kappa}$, $\underline{E}$ and $\underline{B}$ are known from the above method, the charge density $\rho$ and current density $\underline{J}$ can be found.

Finally the scalar potential can be found from:

$$\underline{E} = -\underline{\nabla} \phi + \underline{\omega} \phi = -\frac{\partial \underline{A}}{\partial t} - \omega_0 \underline{A} \quad -(57)$$

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The homogeneous plane wave solutions are:

$$\underline{E} = \frac{E^{(0)}}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right)e^{i\phi} \quad -(58)$$

$$\underline{B} = \frac{B^{(0)}}{\sqrt{2}}\left(i\underline{i} + \underline{j}\right)e^{i\phi} \quad -(59)$$

$$\underline{A} = \frac{A^{(0)}}{\sqrt{2}}\left(i\underline{i} + \underline{j}\right)e^{i\phi} \quad -(60)$$

where

$$\phi = \omega t - \kappa Z \quad -(61)$$

These are solutions of:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(62)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(63)$$

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad -(64)$$

<u>Check</u>

$$\underline{\nabla} \times \underline{A} = \frac{A^{(0)}}{\sqrt{2}}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial X & \partial/\partial Y & \partial/\partial Z \\ ie^{i\phi} & e^{i\phi} & 0 \end{vmatrix} \quad -(65)$$

$$= \frac{A^{(0)}}{\sqrt{2}}\left(-\underline{i}\frac{\partial}{\partial Z}e^{i\phi} + i\underline{j}\frac{\partial}{\partial Z}e^{i\phi}\right)$$

where

$$\frac{\partial}{\partial Z}e^{i\phi} = -i\kappa e^{i\phi} \quad -(66)$$

So

$$\underline{\nabla} \times \underline{A} = \frac{\kappa A^{(0)}}{\sqrt{2}}\left(i\underline{i} + \underline{j}\right)e^{i\phi} \quad -(67)$$

Finally use:

$$B^{(0)} = \kappa A^{(0)} \quad -(68)$$

So

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad \text{QED} \quad -(69)$$

8) Also:

$$\underline{\nabla} \times \underline{E} = \frac{E^{(0)}}{\sqrt{2}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial Z} \\ e^{i\phi} & -ie^{i\phi} & 0 \end{vmatrix}$$

$$= \frac{E^{(0)}}{\sqrt{2}} \left( -i \frac{\partial}{\partial Z} e^{i\phi} \underline{i} + \frac{\partial}{\partial Z} e^{i\phi} \underline{j} \right)$$

$$= \frac{E^{(0)} \kappa}{\sqrt{2}} \left( \underline{i} - i \underline{j} \right) e^{i\phi} \quad -(70)$$

$$\frac{\partial \underline{B}}{\partial t} = i\omega \frac{B^{(0)}}{\sqrt{2}} \left( i\underline{i} + \underline{j} \right) e^{i\phi} \quad -(71)$$

$$= \frac{\omega B^{(0)}}{\sqrt{2}} \left( -\underline{i} + i \underline{j} \right) e^{i\phi}$$

Eq. (63) follows if:

$$E^{(0)} \kappa = \omega B^{(0)} \quad -(72)$$

where

$$E^{(0)} = c B^{(0)} \quad -(73)$$

so

$$\kappa = \frac{\omega}{c} \quad -(74)$$

The spin connection vector is the plane wave:

$$\underline{\omega} = \frac{\omega^{(0)}}{\sqrt{2}} \left( -i\underline{i} + \underline{j} \right) e^{-i\phi} \quad -(75)$$

so the antisymmetry equation (38) is true:

$$-\omega_x A_Y = \omega_Y A_x \quad -(76)$$

Q.E.D.