377(8): Summary of Equations, Calculation of κ_x and κ_y.

Newtonian Limit

The force equation is:

$$\underline{F} = m\underline{g} = -mMG\frac{\underline{r}}{r^3} \quad -(1)$$

and the relevant field equation is:

$$\underline{\nabla}\cdot\underline{g} = \underline{\kappa}\cdot\underline{g} = 4\pi G\rho_m \quad -(2)$$

in previous notation. In component format:

$$\ddot{X} = g_X \quad -(3)$$

$$\ddot{Y} = g_Y \quad -(4)$$

where

$$g_X = -\frac{MGX}{(X^2+Y^2)^{3/2}}; \quad g_Y = -\frac{MGY}{(X^2+Y^2)^{3/2}} \quad -(5)$$

The field equation gives:

$$\frac{\partial g_X}{\partial X} + \frac{\partial g_Y}{\partial Y} = \kappa_X g_X + \kappa_Y g_Y = 4\pi G\rho_m \quad -(6)$$

$$= \frac{MG}{(X^2+Y^2)^{3/2}}$$

So

$$X\kappa_X + Y\kappa_Y = -1. \quad -(7)$$

It follows that

$$\kappa_X = -(1+Y\kappa_Y)/X \quad -(8)$$

$$\kappa_Y = -(1+X\kappa_X)/Y \quad -(9)$$

Therefore X and Y can be found from eqs. (3) and (4), using numerical integration, and

2) K_x plotted against K_y. This is a description of the orbit in terms of K_x and K_y in the Newtonian limit.

The orbit is the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(10)$$

where a is the semi major axis and b is the semi minor axis.

Retrograde Precession

In this case:

$$\ddot{x} = -MG\left(1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2}\right)^{3/2} x \quad -(11)$$

$$\ddot{y} = -MG\left(1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2}\right)^{3/2} y \quad -(12)$$

and the same equation (7) holds with different spin connections. Therefore K_x can be plotted against K_y for retrograde precession, because x, y, $\ddot{x}$ and $\ddot{y}$ are known from the numerical integration of eqns. (11) and (12). The plot of K_x versus K_y will reflect the existence of the retrograde precession. There are no adjustable variables because x, y, $\dot{x}$, $\dot{y}$, $\ddot{x}$ and $\ddot{y}$ can be found from numerical integration.

) Use of the ECE2 Faraday law of induction for gravitostatics gives:

$$\underline{\nabla} \times \underline{g} = \underline{\kappa} \times \underline{g} \quad -(13)$$

$$= \underline{0}$$

so

$$\underline{\kappa} \parallel \underline{g} \quad -(14)$$

From eq. (5) in the Newtonian limit:

$$\kappa_X = - \frac{AMGX}{(X^2+Y^2)^{3/2}} \quad -(15)$$

$$\kappa_Y = - \frac{BMGY}{(X^2+Y^2)^{3/2}} \quad -(16)$$

where A and B are proportionality constants. The units of κ_X and κ_Y are inverse metres and those of g_X and g_Y are $m s^{-2}$. We have:

$$\kappa_X = A g_X \quad -(17)$$

$$\kappa_Y = B g_Y \quad -(18)$$

so the units of A and B are $s^2 m^{-2}$, the units of inverse square velocity. If it is assumed that this is a constant, then:

$$\kappa_X = - \frac{MGX}{v_0^2 (X^2+Y^2)^{3/2}} \quad -(19)$$

$$\kappa_Y = - \frac{MGY}{v_0^2 (X^2+Y^2)^{3/2}} \quad -(20)$$

4) Units Check

$$mG = m^3 s^{-2}, \quad g = m^3 s^{-2} / m^2 = m s^{-2}$$ ✓

$$A = B = m^{-2} s^2, \quad g = 1/(m^2 s^{-2})$$ ✓

Now use:

$$\underline{\nabla} \cdot \underline{g} = 4\pi G \rho_m = \underline{\kappa} \cdot \underline{g} \quad -(22)$$

From eqs. (19), (20), (5), (6) and (22):

$$\kappa_X g_X + \kappa_Y g_Y = 4\pi G \rho_m \quad -(23)$$

and

$$X \kappa_X + Y \kappa_Y = -1 \quad -(24)$$

From eqs. (19), (20) and (24):

$$\frac{mG(X^2+Y^2)}{v_0^2 (X^2+Y^2)^{3/2}} = 1 \quad -(25)$$

and

$$\boxed{v_0^2 = \frac{mG}{(X^2+Y^2)^{1/2}} = \frac{mG}{r}} \quad -(26)$$

From eqs. (19), (20) and (26):

$$\boxed{\begin{aligned} \kappa_X &= -\frac{X}{X^2+Y^2} \quad -(27) \\ \kappa_Y &= -\frac{Y}{X^2+Y^2} \quad -(28) \end{aligned}}$$

in the Newtonian limit

5) Spin Connections for Retrograde Precession

In this case:

$$g_X = -\frac{mGX}{\gamma^3(X^2+Y^2)^{3/2}} \quad -(29)$$

$$g_Y = -\frac{mGY}{\gamma^3(X^2+Y^2)^{3/2}} \quad -(30)$$

so:

$$\kappa_X = -\frac{mGX}{\gamma^3 v_0^2(X^2+Y^2)^{3/2}} \quad -(31)$$

$$\kappa_Y = -\frac{mGY}{\gamma^3 v_0^2(X^2+Y^2)^{3/2}} \quad -(32)$$

From eqs. (31), (32) and (24):

$$v_0^2 = \frac{1}{\gamma^3}\frac{mG}{(X^2+Y^2)^{1/2}} = \frac{mG}{\gamma^3 r} \quad -(33)$$

so the same result is obtained:

$$\kappa_X = -\frac{X}{X^2+Y^2} \quad -(34)$$

$$\kappa_Y = -\frac{X}{X^2+Y^2} \quad -(35)$$

but X and Y are different in the precessing orbit. The orbit is completely determined by κ_X and κ_Y, given the force and field equations of ECE 2.