377(7): Interpretation of the Precession Formulae

From Eq. (6) of Note 377(5):

$$\frac{\dot{x}}{x}(x\dot{x}+y\dot{y}) = \frac{\dot{y}}{y}(y\dot{y}+x\dot{x}) \quad -(1)$$

So:

$$\boxed{\dot{x}y = x\dot{y}} \quad -(2)$$

Under condition (2), Eqs. (1) and (2) of Note 377(5) follow.

Consider the equation of an ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1. \quad -(3)$$

Differentiating with respect to time:

$$\frac{2x\dot{x}}{a^2}+\frac{2y\dot{y}}{b^2} = 0 \quad -(4)$$

so:

$$b^2x\dot{x}+a^2y\dot{y} = 0 \quad -(5)$$

i.e.

$$b^2x\dot{x} = -a^2y\dot{y} \quad -(6)$$

Multiply Eq. (6) by $\dot{x}$:

$$a^2\dot{x}y\dot{y} = -b^2x\dot{x}^2 \quad -(7)$$

So:

$$\dot{x}y = -\frac{b^2}{a^2}\frac{x\dot{x}^2}{\dot{y}} \quad -(8)$$

Multiply Eq (6) by x:

$$a^2xy\dot{y} = -b^2x^2\dot{x} \quad -(9)$$

and
$$x\dot{y} = -\frac{b^2}{a^2}\frac{x^2\dot{x}}{y} \quad -(10)$$

So Eq. (2) implies:
$$\frac{x\dot{x}^2}{\dot{y}} = \frac{x^2\dot{x}}{y} \quad -(11)$$

i.e.
$$xy\dot{x}^2 = x^2\dot{x}\dot{y}^2 \quad -(12)$$

or
$$y\dot{x} = x\dot{y} \quad -(13)$$

Q.E.D.

Therefore the condition (2) is satisfied by a static ellipse. Under this condition the forward and retrograde precessions are the same, and this can only be true if they are both zero, i.e. the ellipse is static.

This result means that the two methods of showing a precessing ellipse are fundamentally different once relativistic effects are taken into consideration. As shown in Note 377(4), they are the same if and only if:
$$\gamma \rightarrow 1. \quad -(14)$$

Furthermore, the non-relativistic limit defined in Note 377(6) gives a circle, because
$$x\dot{x} + y\dot{y} = 0 \quad -(15)$$
true only for a circle:

$$x^2 + y^2 = r_1^2 = \text{constant} \quad - (16)$$

Differentiating eq. (16) gives:

$$2(x\dot{x} + y\dot{y}) = 0 \quad - (17)$$

from which eq. (7) follows, Q.E.D.

Conclusions

1) Forward precessions are defined by:

$$\ddot{x} = \frac{mG}{\gamma^2 (x^2+y^2)^{3/2}} \left(\frac{\dot{x}\dot{y}y + x\dot{x}}{c^2} - x \right) \quad - (18)$$

$$\ddot{y} = \frac{mG}{\gamma^2 (x^2+y^2)^{3/2}} \left(\frac{\dot{y}\dot{x}x + y\dot{y}^2}{c^2} - y \right) \quad - (19)$$

where

$$\gamma = \left(1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2} \right)^{-1/2} \quad - (20)$$

2) Retrograde precessions are defined by:

$$\ddot{x} = - \frac{mGx}{(x^2+y^2)^{3/2}\gamma^3} \quad - (21)$$

$$\ddot{y} = \frac{-mGy}{(x^2+y^2)^{3/2}\gamma^3} \quad - (22)$$

In both cases the acceleration due to gravity is:

$$\underline{g} = \ddot{x}\,\underline{i} + \ddot{y}\,\underline{j} \quad - (23)$$

and is governed by the field equations of ECE2 gravitostatics:

$$\underline{\nabla} \times \underline{g} = \underline{0} \quad -(24)$$

and

$$\underline{\nabla} \cdot \underline{g} = \underline{\kappa} \cdot \underline{g} = 4\pi G \rho_m \quad -(25)$$

where ρ_m is the source mass density. Therefore:

$$\frac{\partial \ddot{X}}{\partial X} + \frac{\partial \ddot{Y}}{\partial Y} = \kappa_X \ddot{X} + \kappa_Y \ddot{Y} \quad -(26)$$

$$= 4\pi G \rho_m$$

If for the sake of argument:

$$\frac{\partial \ddot{X}}{\partial X} = \kappa_X \ddot{X} \quad -(27)$$

and

$$\frac{\partial \ddot{Y}}{\partial Y} = \kappa_Y \ddot{Y} \quad -(28)$$

then κ_X and κ_Y are determined from eqns (18) to (20) or eqns. (21) and (22).

Forward and retrograde precessions give different spin connections. The observed precession is determined by the spin connection.

However, the spin connections κ_X and κ_Y can be regarded as input parameters that determine the boundary and initial conditions for eqns. (18) to (20) or Eqns. (21) and (22). For example:

$$\frac{\partial \ddot{X}(0)}{\partial X} + \frac{\partial \ddot{Y}(0)}{\partial Y} = \kappa_X \ddot{X}(0) + \kappa_Y \ddot{Y}(0) \quad -(29)$$

) and

$$\frac{\partial \ddot{X}(0)}{\partial X} = k_X \ddot{X}(0) \quad - (30)$$

$$\frac{\partial \ddot{Y}(0)}{\partial Y} = k_Y \ddot{Y}(0) \quad - (31)$$

The orbits and precessions are different for different initial conditions. Integration of eqs. (30) and (31) give $X(0)$, $Y(0)$, $\dot{X}(0)$ and $\dot{Y}(0)$ at some point such as the perihelion. Finally k_X and k_Y are adjusted to give the observed precession, for example of planets in the solar system or an S2 star system (retrograde precession).
