77(6): Constraint Equation in the Non-relativistic Limit

In this limit:

$$\ddot{X} = \frac{mG}{\gamma^2(X^2+Y^2)^{3/2}}\left(\frac{\dot{X}\dot{Y}Y + X\dot{X}^2}{c^2} - X\right) = -\frac{mGX}{\gamma^3(X^2+Y^2)^{3/2}} \quad -(1)$$

$$\ddot{Y} = \frac{mG}{\gamma^2(X^2+Y^2)^{3/2}}\left(\frac{\dot{Y}\dot{X}X + Y\dot{Y}^2}{c^2} - Y\right) = \frac{-mGY}{\gamma^3(X^2+Y^2)^{3/2}} \quad -(2)$$

reduce to:

$$\ddot{X} = -\frac{mGX}{(X^2+Y^2)^{3/2}} \quad -(3)$$

$$\ddot{Y} = -\frac{mGY}{(X^2+Y^2)^{3/2}} \quad -(4)$$

So:

$$\dot{X}\dot{Y}Y + X\dot{X}^2 = \dot{Y}\dot{X}X + Y\dot{Y}^2 = 0 \quad -(5).$$

and

$$\gamma \rightarrow 1 \quad -(6)$$

Eq. (5) means that:

$$X\dot{X} + Y\dot{Y} = 0 \quad -(7)$$

and

$$\frac{X}{Y} = -\frac{\dot{Y}}{\dot{X}} \quad -(8)$$

i.e.

$$\frac{X^2}{Y^2} = \frac{\dot{Y}^2}{\dot{X}^2} \quad -(9)$$

This is a particular solution of eq. (10) of Note 377(5):

$$\frac{\dot{X}}{\dot{Y}} - \frac{\dot{Y}}{\dot{X}} = \frac{X}{Y} - \frac{Y}{X} \quad -(10)$$

In the limit (5) and (1) the orbit is the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(11)$$

where a is the semi major and b is the semi minor axis

$$b^2 x^2 + a^2 y^2 = a^2 b^2 \quad -(12)$$

so

$$x^2 = a^2 \left(1 - \frac{y^2}{b^2} \right) \quad -(13)$$

and

In polar representation:

$$x = a\epsilon + r\cos\phi \quad -(14)$$

$$y = r\sin\phi \quad -(15)$$

where ϵ is the eccentricity. The velocity is:

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2 \quad -(16)$$

From eqs. (14) and (15):

$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \quad -(17)$$

$$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \quad -(18)$$

So:

$$x\dot{x} + y\dot{y} = a\epsilon(\dot{r}\cos\phi - r\dot{\phi}\sin\phi)$$
$$+ r\cos\phi(\dot{r}\cos\phi - r\dot{\phi}\sin\phi) \quad -(19)$$
$$+ r\sin\phi(\dot{r}\sin\phi + r\dot{\phi}\cos\phi)$$
$$= a\epsilon(\dot{r}\cos\phi - r\dot{\phi}\sin\phi) + r\dot{r} = 0$$

The constraint equation is therefore:

$$\boxed{r\dot{r} + a\epsilon\left(\dot{r}\cos\phi - r\dot{\phi}\sin\phi\right) = 0} \quad -(20)$$

in the non-relativistic limit. In the case of a circular orbit:

$$\dot{r} = \epsilon = 0 \quad -(21)$$

so eq. (20) is self consistent.

Eqs. (9) and (13) give the ratio of $\dot{y}^2$ to $\dot{x}^2$ in terms of x^2/y^2, where:

$$x^2 = \frac{1}{b^2}\left(a^2b^2 - a^2y^2\right) \quad -(22)$$

So

$$\frac{x^2}{y^2} = \frac{a^2}{y^2} - \frac{a^2}{b^2} = a^2\left(\frac{1}{y^2} - \frac{1}{b^2}\right) \quad -(23)$$

and

$$\frac{\dot{y}^2}{\dot{x}^2} = a^2\left(\frac{1}{y^2} - \frac{1}{b^2}\right) \quad -(24)$$

Therefore:

$$v^2 = \dot{x}^2 + \dot{y}^2 \quad -(25)$$

is affected by the constraint condition. From eqs. (8), (14) and (15):

$$\frac{\dot{y}}{\dot{x}} \underset{(7)}{=} -\frac{x}{y} = -\left(\frac{a\epsilon + r\cos\phi}{r\sin\phi}\right) \quad -(26)$$

so the constraint gives the ratio of $\dot{y}$ to $\dot{x}$ in terms of the orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(27)$$