(4): Comparison of Euler Lagrange Equations

ethod One

The Lagrangian is:

$$\mathcal{L} = -mc^2\left(1 - \frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{1/2} + \frac{mMG}{(x^2+y^2)^{1/2}} \quad -(1)$$

he proper Lagrange variables are X and Y, and the Euler agrange equations are:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \quad -(2)$$

nd

$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}} \quad -(3)$$

These are scalar Euler Lagrange equations.

By computer algebra they give:

$$\ddot{X} = \frac{mG}{\gamma^2(X^2+Y^2)^{3/2}}\left(\frac{\dot{X}\dot{Y}Y + X\dot{X}^2}{c^2} - X\right) \quad -(4)$$

and

$$\dot{Y} = \frac{mG}{\gamma^2(X^2+Y^2)^{3/2}}\left(\frac{\dot{Y}\dot{X}X + Y\dot{Y}^2}{c^2} - Y\right) \quad -(5)$$

here:

$$\gamma = \left(1 - \frac{\dot{X}^2+\dot{Y}^2}{c^2}\right)^{-1/2} \quad -(6)$$

When solved they give:

$$\Delta\phi = 5.903 \times 10^{-4} \text{ rad} \quad -(7)$$

for an initial condition of:

$$v(0) = 7.7529 \times 10^6 \ ms^{-1}$$

i.e. they give a precessing orbit in a plane with forward precession.

Method Two

The lagrangian is:

$$\mathcal{L} = -mc^2\left(1 - \frac{\dot{\underline{r}} \cdot \dot{\underline{r}}}{c^2}\right)^{1/2} + \frac{mMG}{|\underline{r}|} \quad -(8)$$

where

$$\underline{r} = X\underline{i} + Y\underline{j} \quad -(9)$$

The proper Lagrange variable is $\underline{r}$, and the Euler Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} \quad -(10)$$

where

$$\underline{p} = \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} \quad -(11)$$

is the relativistic momentum:

$$\underline{p} = \gamma m \underline{v}_0 \quad -(12)$$

here

$$\underline{v}_0 = \dot{\underline{r}}. \quad -(13)$$

In Eq. (10):

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} = \frac{d\underline{p}}{dt} = \gamma^3 m \ddot{\underline{r}} \quad -(14)$$

) and

$$\frac{\partial \mathcal{L}}{\partial \mathbf{r}} = -mMG\frac{\mathbf{r}}{r^3} \quad -(15)$$

so the Euler Lagrange equation (10) gives the relativistic planar orbital equation:

$$\gamma^3 \ddot{\mathbf{r}} = -MG\frac{\mathbf{r}}{r^3} \quad -(16)$$

i.e.

$$\ddot{X} = -\frac{MG}{\gamma^3 (X^2+Y^2)^{3/2}} X \quad -(17)$$

and

$$\ddot{Y} = -\frac{MG}{\gamma^3 (X^2+Y^2)^{3/2}} Y \quad -(18)$$

When solved these give a negative or retrograde precession of:

$$\Delta\phi = -1.7697 \times 10^{-3} \text{ radians}$$

for an initial condition of

$$v(0) = 7.7529 \times 10^{6} \, m s^{-1} \quad -(19)$$

Eq. (10) is the vector Euler Lagrange equation.

The experimental result for the S2 star system is between -0.017 and $+0.035$ radians. The theoretical results are

$$\Delta\phi = 5.903 \times 10^{-4} \text{ rad. (method one)} \quad -(20)$$

$$\Delta\phi = -1.7697 \times 10^{-3} \text{ rad (method two)} \quad -(21)$$

The theoretical results are in the middle of the experimental range.

4) In method one, the relativistic angular momentum is conserved. In method two the non-relativistic angular momentum is conserved.

The choice of lagrangian and Euler Lagrange variables is all important. The scalar and vector Euler Lagrange equations give positive and negative precession respectively in an ECE2 covariant theory.

<u>Non Relativistic Limit</u>

<u>Method One</u>

The non-relativistic lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{X}^2 + \dot{Y}^2 \right) + \frac{mMG}{(X^2+Y^2)^{1/2}} \quad -(22)$$

The scalar Euler Lagrange equations (2) and (3) give:

$$m\ddot{X} = mMG \frac{d}{dX} \left(\frac{1}{(X^2+Y^2)^{1/2}} \right)$$

$$= -mMG \frac{X}{(X^2+Y^2)^{3/2}} \quad -(23)$$

and

$$m\ddot{Y} = -mMG \frac{Y}{(X^2+Y^2)^{3/2}} \quad -(24)$$

which give the static elliptical orbit <u>with no precession</u>.

<u>Method Two</u>

The non relativistic lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \, \dot{\underline{r}} \cdot \dot{\underline{r}} + \frac{mMG}{|\underline{r}|} \quad -(25)$$

and the vector Euler Lagrange equation (10)

) gives:

$$m\ddot{\underline{r}} = -mMG\frac{\underline{r}}{r^3} \quad -(26)$$

which is the vector form of eqs. (23) and (24), Q.E.D.

So in the Newtonian or non-relativistic theory of planar orbits, both methods give the same result. In the relativistic theory they give different results.

The vector Lagrangian is the same as the scalar Lagrangian because:

$$\dot{\underline{r}} \cdot \dot{\underline{r}} = \dot{X}^2 + \dot{Y}^2 \quad -(27)$$

and

$$\frac{1}{|\underline{r}|} = \frac{1}{r} = \frac{1}{(X^2+Y^2)^{1/2}} \quad -(28)$$

However, the vector Euler Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial(X\underline{i}+Y\underline{j})} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\dot{X}\underline{i}+\dot{Y}\underline{j})} \quad -(29)$$

Now note that:

$$\frac{\partial \mathcal{L}}{\partial(X\underline{i}+Y\underline{j})} = \frac{\partial \mathcal{L}}{\partial X}\frac{\partial X}{\partial(X\underline{i}+Y\underline{j})} = \frac{1}{\underline{i}}\frac{\partial \mathcal{L}}{\partial X} \quad -(30)$$

so

$$\frac{\partial \mathcal{L}}{\partial X} = \underline{i} \cdot \frac{\partial \mathcal{L}}{\partial(X\underline{i}+Y\underline{j})} \quad -(31)$$

Similarly:

$$\frac{\partial \mathcal{L}}{\partial Y} = \underline{j} \cdot \frac{\partial \mathcal{L}}{\partial(X\underline{i}+Y\underline{j})} \quad -(32)$$

Also:

$$\frac{\partial \mathcal{L}}{\partial \dot{X}} = \underline{i} \cdot \frac{\partial \mathcal{L}}{\partial(\dot{X}\underline{i}+\dot{Y}\underline{j})} \quad -(33)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{Y}} = \underline{j} \cdot \frac{\partial \mathcal{L}}{\partial(\dot{X}\underline{i} + \dot{Y}\underline{j})} \quad -(34)$$

Therefore:

$$\underline{i} \cdot \frac{\partial \mathcal{L}}{\partial(X\underline{i} + Y\underline{j})} = \underline{i} \cdot \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\dot{X}\underline{i} + \dot{Y}\underline{j})} \quad -(35)$$

gives

$$\frac{\partial}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}}, \quad -(36)$$

and

$$\underline{j} \cdot \frac{\partial \mathcal{L}}{\partial(X\underline{i} + Y\underline{j})} = \underline{j} \cdot \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\dot{X}\underline{i} + \dot{Y}\underline{j})} \quad -(37)$$

gives

$$\frac{\partial}{\partial Y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}}. \quad -(38)$$

The scalar Euler Lagrange equations are the two components of the vector Euler Lagrange equation. The system is equivalent, so it is concluded that forward precessions are accompanied by retrograde precessions.