377(1): Orbital Equation with Relativistic Form of Newton's Law

In this case the force is defined by

$$\underline{F} = \frac{d}{dt}\left(\gamma m \underline{v}_0\right) = \frac{dp}{dt} \quad -(1)$$

The relativistic kinetic energy is derived from this force:

$$W = T = \int \frac{d}{dt}\left(\gamma m \underline{v}_0\right)\cdot v_0\, dt \quad -(2)$$

where W is the work done. So:

$$T = m\int_0^{v_0} v_0\, d\left(\gamma v_0\right) \quad -(3)$$

and integrating by parts:

$$T = \gamma m v_0^2 - m\int_0^{v_0} \frac{v_0\, dv_0}{\left(1 - v_0^2/c^2\right)^{1/2}} \quad -(4)$$

$$= \gamma m v_0^2 + mc^2\left(1 - v_0^2/c^2\right)\Big|_0^{v_0}$$

$$= \gamma m v_0^2 + mc^2\left(1 - v_0^2/c^2\right)^{1/2} - mc^2$$

$$= (\gamma - 1)mc^2$$

This procedure gives the total energy:

$$E = T + mc^2 = \gamma mc^2 \quad -(5)$$

and the hamiltonian:

$$H = \gamma mc^2 + U \quad -(6)$$

and the lagrangian:

2)

$$\mathcal{L} = -mc^2\left(1-\frac{v_0^2}{c^2}\right)^{-1/2} - U \quad -(7)$$

$$= -\frac{mc^2}{\gamma} - U \quad -(8)$$

So the force that corresponds to eqs. (6) and (8) is:

$$F = \frac{d}{dt}(\gamma m v_0) = m\frac{d}{dt}(\gamma v_0)$$

$$= m\left(v_0\frac{d\gamma}{dt} + \gamma\frac{dv_0}{dt}\right)$$

$$= m\left(v_0\frac{d\gamma}{dv_0}\frac{dv_0}{dt} + \gamma\frac{dv_0}{dt}\right)$$

$$= m\frac{dv_0}{dt}\left(v_0\frac{d\gamma}{dv_0} + \gamma\right) \quad -(9)$$

Now use: $$\gamma = \left(1-\frac{v_0^2}{c^2}\right)^{-1/2} \quad -(10)$$

so $$\frac{d\gamma}{dv_0} = \frac{v_0}{c^2}\left(1-\frac{v_0^2}{c^2}\right)^{-3/2} = \gamma^3\frac{v_0}{c^2} \quad -(11)$$

So $$F = m\gamma\frac{dv_0}{dt}\left(1+\gamma^2\frac{v_0^2}{c^2}\right) \quad -(12)$$

$$= m\gamma\frac{dv_0}{dt}\left(1+\left(1-\frac{v_0^2}{c^2}\right)^{-1}\frac{v_0^2}{c^2}\right)$$

3)

$$= m\gamma \frac{dv_0}{dt}\left(1 + \frac{v_0^2}{c^2\left(1 - \frac{v_0^2}{c^2}\right)}\right)$$

$$= m\gamma \frac{dv_0}{dt}\frac{\left(c^2\left(1-\frac{v_0^2}{c^2}\right) + v_0^2\right)}{c^2\left(1-\frac{v_0^2}{c^2}\right)}$$

$$= m\gamma^3 \frac{dv_0}{dt}$$

Therefore the orbital equation is:

$$\gamma^3 \ddot{\underline{r}} = -mG\frac{\underline{r}}{r^3} \quad -(13)$$

i.e.

$$\ddot{X} = -mG\left(1 - \frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{3/2}\frac{X}{(x^2+y^2)^{3/2}} \quad -(14)$$

$$\ddot{Y} = -mG\left(1 - \frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{3/2}\frac{Y}{(x^2+y^2)^{3/2}} \quad -(15)$$

The components of acceleration are:

$$g_x = \frac{\ddot{X}}{\left(1 + \frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{3/2}} = -mG\frac{X}{(x^2+y^2)^{3/2}} \quad -(16)$$

and
$$g_Y = \frac{\ddot{Y}}{\left(1+\frac{\dot{X}^2+\dot{Y}^2}{c^2}\right)^{3/2}} = -\frac{mG\,Y}{(X^2+Y^2)^{3/2}} \quad -(17)$$

The relevant field equations are:

$$\underline{\nabla} \times \underline{g} = \underline{0} \quad -(18)$$

$$\underline{\nabla} \cdot \underline{g} = \underline{\kappa} \cdot \underline{g} = 4\pi G \rho_m \quad -(19)$$

$$\frac{\partial \underline{g}}{\partial t} = \underline{0} \quad -(20)$$

Eq. (19) gives:

$$\rho_m = \frac{m}{2\pi (X^2+Y^2)^{3/2}} \quad -(21)$$

and
$$\frac{\partial g_X}{\partial X} + \frac{\partial g_Y}{\partial Y} = \frac{2mG}{(X^2+Y^2)^{3/2}} \quad -(22)$$

$$= \kappa_X g_X + \kappa_Y g_Y$$

$$= \frac{\kappa_X \ddot{X} + \kappa_Y \ddot{Y}}{\left(1+\frac{\dot{X}^2+\dot{Y}^2}{c^2}\right)^{3/2}}$$

$$= -mG\left(\frac{\kappa_X X + \kappa_Y Y}{(X^2+Y^2)^{3/2}}\right)$$

So there are two more equations:

$$X\kappa_X + Y\kappa_Y = -2 \quad -(23)$$

and

$$\kappa_x g_x + \kappa_y g_y = 4\pi G \rho_m = \frac{\kappa_x \ddot{X} + \kappa_y \ddot{Y}}{\left(1 + \frac{\dot{X}^2 + \dot{Y}^2}{c^2}\right)^{3/2}} \quad -(24)$$

Eqs. (14), (15), (23) and (24) must be solved simultaneously with input parameters κ_x and κ_y.
Note that eqs. (14) and (15) are possible solutions of eq. (24), the solutions:

$$\kappa_x g_x = \frac{\kappa_x \ddot{X}}{\left(1 + \frac{\dot{X}^2 + \dot{Y}^2}{c^2}\right)^{3/2}} \quad -(25)$$

and

$$\kappa_y g_y = \frac{\kappa_y \ddot{Y}}{\left(1 + \frac{\dot{X}^2 + \dot{Y}^2}{c^2}\right)^{3/2}} \quad -(26)$$

However, a more general solution is found from eq. (24) by using κ_x and κ_y as input parameters. Eq. (24) is then solved simultaneously with eqs. (14) and (15) to give all kinds of precessing orbits.

The Minkowski force is:

$$F^\mu = dp^\mu / d\tau \quad -(27)$$

where

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right) \quad -(28)$$

) is the relativistic four momentum. Here:

$$E = \gamma m c^2, \quad p = \gamma m \underline{v}. \quad -(29)$$

The vector part of the Minkowski four-force is:

$$F\,(\text{Minkowski}) = \gamma \underline{F} \quad -(30)$$

where $\underline{F}$ is defined in eq. (1).

It is clear that eq. (1) is the force definition that leads to the lagrangian (8) and hamiltonian (7). So for self consistency the orbital force equation is (13). The Minkowski four force is defined with the proper time, i.e. the time in the frame which the particle's own time, attached to the particle. Therefore eq. (13) in the particle frame is:

$$\gamma^4 \ddot{\underline{r}} = -MG\frac{\underline{r}}{r^3}. \quad -(31)$$

In the laboratory or observer frame, the equations to be solved are:

$$\gamma^3 \ddot{\underline{r}} = -MG\frac{\underline{r}}{r^3} \quad -(32)$$

together with the field equations in the laboratory or observer frame, eqs. (18) to (20).