375(8): Relativistic Lagrangian for the Binary Pulsar

If the masses of the two stars are m_1 and m_2, then with reference to Fig (1):

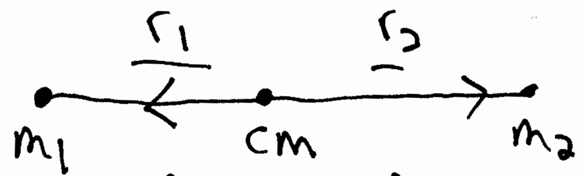

Fig (1)

where CM denotes the centre of mass, the non-relativistic Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1 + \frac{1}{2} m_2 \dot{\underline{r}}_2 \cdot \dot{\underline{r}}_2 + \frac{m_1 m_2 G}{r} \quad -(1)$$

where

$$r = |\underline{r}_1 - \underline{r}_2| = \left((\underline{r}_1 - \underline{r}_2) \cdot (\underline{r}_1 - \underline{r}_2) \right)^{1/2} \quad -(2)$$

In Cartesian coordinates:

$$\underline{r}_1 = X_1 \underline{i} + Y_1 \underline{j} \quad -(3)$$

$$\underline{r}_2 = X_2 \underline{i} + Y_2 \underline{j} \quad -(4)$$

So:

$$\mathcal{L} = \frac{1}{2} m_1 \left(\dot{X}_1^2 + \dot{Y}_1^2 \right) + \frac{1}{2} m_2 \left(\dot{X}_2^2 + \dot{Y}_2^2 \right) + \frac{m_1 m_2 G}{r} \quad -(5)$$

where

$$r = \left((X_1 - X_2)^2 + (Y_1 - Y_2)^2 \right)^{1/2} \quad -(6)$$

The proper Lagrange variables are X_1, Y_1, X_2 and Y_2.

So:

$$\frac{\partial \mathcal{L}}{\partial X_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}_1} \quad -(7)$$

$$\frac{\partial \mathcal{L}}{\partial Y_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}_1} \quad -(8)$$

$$\frac{\partial \mathcal{L}}{\partial X_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}_2} \quad -(9)$$

$$\frac{\partial \mathcal{L}}{\partial Y_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}_2} \quad -(10)$$

These four simultaneous equations can be integrated numerically to give the orbits of m_1 and m_2 around the centre of mass. These are ellipses with the centre of mass at one focus of the ellipse. This calculation checks the integration code.

The earth sun system is described by:

$$m_2\,(\text{sun}) \gg m_1\,(\text{earth}) \quad -(11)$$

and

$$\underline{r}_1 \gg \underline{r}_2 \quad -(12)$$

so the centre of mass is essentially the position of the sun. Then eq. (5) reduces to:

$$\mathcal{L}(\text{earth}/\text{sun}) = \frac{1}{2} m_1 \left(\dot{X}_1^2 + \dot{Y}_1^2 \right) + \frac{m_1 m_2 G}{\left(X_1^2 + Y_1^2 \right)^{1/2}} \quad -(13)$$

with the Euler Lagrange equations (7) and (8). Eqs. (7), (8) and (13) give an ellipse with m_2 at one focus.

3) This ellipse is, in plane polar coordinates:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(14)$$

where

$$\alpha = a(1-\epsilon^2) \quad -(15)$$

Here a is the semi major axis, ϵ the eccentricity and α the half right latitude. If b is the semi minor axis then

$$\alpha = b(1-\epsilon^2)^{1/2} \quad -(16)$$

The ellipse from eqs. (7), (8) and (13) should obey:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(17)$$

where:

$$x = c + r\cos\phi \quad -(18)$$
$$y = r\sin\phi$$

and

$$\epsilon = \left(1 - \frac{b^2}{a^2}\right)^{1/2} = \frac{c}{a}. \quad -(19)$$

The perihelion of the ellipse is defined by:

$$\phi = 0, \quad \cos\phi = 1 \quad -(20)$$

so

$$r_{min} = \frac{\alpha}{1+\epsilon} \quad -(21)$$

The orbital velocity at the perihelion is:

$$v^2 = m_2 G\left(\frac{2}{r_{min}} - \frac{1}{a}\right) \quad -(22)$$

$$= \left(\dot{x}_1^2 + \dot{y}_1^2\right)$$

Therefore at the perihelion:

$$v^2 = \frac{m_2 G}{d}\left(2(1+\epsilon) + 1 - \epsilon^2\right) \quad -(23)$$

So at the perihelion the integration program should produce:

$$\dot{X}_1^2 + \dot{Y}_1^2 = \frac{m_2 G}{d}\left(3 + 2\epsilon - \epsilon^2\right) \quad -(24)$$

Relativistic Lagrangian

The relativistic Lagrangian of the binary pulsar is:

$$\mathcal{L} = -m_1 c^2\left(1 - \frac{\dot{X}_1^2 + \dot{Y}_1^2}{c^2}\right)^{1/2} - m_2 c^2\left(1 - \frac{\dot{X}_2^2 + \dot{Y}_2^2}{c^2}\right)^{1/2} + \frac{m_1 m_2 G}{(X_1^2 + Y_1^2)^{1/2}} \quad -(25)$$

and the Euler Lagrange equations (7) to (10) are solved with this Lagrangian.

In the limit:

$$m_2 \gg m_1 \; ; \; r_1 \gg r_2 \quad -(26)$$

Eq. (25) reduces to:

$$\mathcal{L} = -m_1 c^2\left(1 - \frac{\dot{X}_1^2 + \dot{Y}_1^2}{c^2}\right)^{1/2} + \frac{m_1 m_2 G}{(X_1^2 + Y_1^2)^{1/2}} \quad -(27)$$

5) Eqs. (7), (8) and (27) produce a precessing ellipse, defined by $X_1(t)$, $Y_1(t)$, $\dot{X}_1(t)$ and $\dot{Y}_1(t)$. Similarly eqs. (7) to (10) and (25) produce a combination of precessing ellipses, defined by $X_1(t)$, $Y_1(t)$, $X_2(t)$ and $Y_2(t)$ and $\dot{X}_1(t)$, $\dot{Y}_1(t)$, $\dot{X}_2(t)$ and $\dot{Y}_2(t)$. The precessions are maximized by large masses m_1 and m_2, as in a binary pulsar.

In a static ellipse the function (14) is the same after:

$$\phi \rightarrow \phi + 2\pi \quad -(28)$$

because:

$$\cos\phi = \cos(\phi + 2\pi) \quad -(29)$$

but in a precessing ellipse:

$$\phi \rightarrow \phi + \Delta\phi \quad -(30)$$

after a 2π revolution, where $\Delta\phi$ is the precession per orbit. Therefore in a precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \rightarrow \frac{\alpha}{1+\epsilon\cos(\phi+\Delta\phi)} \quad -(31)$$

after one orbit (a 2π revolution of ϕ). So at the perihelion:

$$r_{min} = \frac{\alpha}{1+\epsilon} \rightarrow \frac{\alpha}{1+\epsilon\cos(2\pi+\Delta\phi)} \quad -(32)$$

) In eq. (32):

$$\cos(2\pi + \Delta\phi) = \cos 2\pi \cos\Delta\phi - \sin(2\pi)\sin\Delta\phi$$
$$= \cos(\Delta\phi) \quad -(33)$$

So after one orbit:

$$r_{min} = \frac{\alpha}{1+\epsilon} \rightarrow \frac{\alpha}{1+\epsilon\cos\Delta\phi} \quad -(34)$$

Suggested Computation Scheme

1) Compute the elliptical orbit from eqns. (7), (8) and (13).
2) Compute the precessing elliptical orbit from eqs. (7), (8) and (27).
3) Measure $\Delta\phi$ graphically and show that the precession increases for a very large $m_2 \gg m_1$
4) Compute the elliptical orbit from eqs. (7) – (10) and Eq. (5). Demonstrate graphically the effect of increasing m_1 in relation to m_2.
5) Compute the precessing elliptical orbit from eqs. (7) to (10) and eq. (25). Show that the precession increases as m_1 and m_2 get heavier.
