375(3): Relativistic Lagrangian in Cartesian Coordinates

This is

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r} \quad -(1)$$

$$= -mc^2\left(1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{1/2} + \frac{mMG}{(x^2+y^2)^{1/2}}$$

Note that the kinetic energy and potential energy satisfy the requirements of the Hamilton Principle of Least Action:

$$T = T(\dot{x}, \dot{y}), \quad U = U(x, y) \quad -(2)$$

so X and Y are proper Lagrange variables and:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right) \quad -(3)$$

$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Y}}\right) \quad -(4)$$

The constant angular momentum is:

$$\underline{L} = \underline{r} \times \underline{p} = m(X\dot{Y} - Y\dot{X})\underline{k} \quad -(5)$$

so

$$X\dot{Y} - Y\dot{X} = \frac{L_Z}{m} = \text{constant} \quad -(6)$$

Therefore eqs. (3), (4) and (6) can be solved for the precessing orbit.