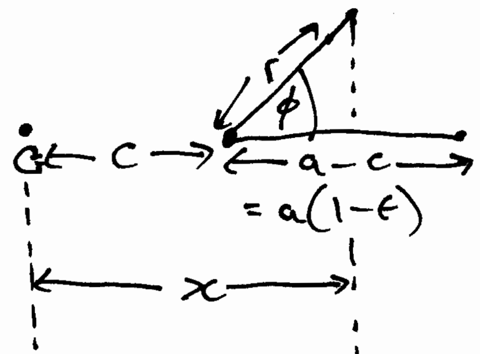
375(2): Derivation of Note 375(1) with Offset Ellipse

The offset ellipse is the orbit of a mass m around a mass M situated at one focus, as in Fig (1).

Fig (1)



This is the orbit observed astronomically. The ellipse is defined by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(1)$$

Also

$$x = c + r\cos\phi \quad -(2)$$

$$y = r\sin\phi \quad -(3)$$

The eccentricity of the ellipse is defined by:

$$\epsilon = \left(1 - \frac{b^2}{a^2}\right)^{1/2} = \frac{c}{a} \quad -(4)$$

Eq. (1) is written as

$$b^2(c^2 + 2cr\cos\phi + r^2\cos^2\phi) + a^2r^2\sin^2\phi = a^2b^2 \quad -(5)$$

From eqs. (4) and (5) it follows that:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(6)$$

2) where

$$\alpha = a(1-\epsilon^2) \quad (7)$$

Note that c is a constant, so:

$$\dot{X} = \frac{d}{dt}\left(c + r\cos\phi(t)\right) \quad (8)$$

$$= \dot{r}\cos\phi - r\dot{\phi}\sin\phi$$

and

$$\dot{Y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \quad (9)$$

and the analysis continues in the same way as in note 375(1), eq. (10) ff. Therefore if:

$$X = c + r\cos\phi \quad (10)$$

$$Y = r\sin\phi \quad (11)$$

and if

$$\ddot{X} = -\frac{mG}{(X^2+Y^2)^{3/2}}X \quad (12)$$

$$\ddot{Y} = -\frac{mG}{(X^2+Y^2)^{3/2}}Y \quad (13)$$

then:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{mG}{r^2} \quad (14)$$

and

$$2\dot{r}\dot{\phi} + r\ddot{\phi} = 0 \quad (15)$$

with

$$\dot{\phi} = \frac{L}{mr^2} \quad (16)$$

Q.E.D.