75(1): Equivalence of Planar Orbital Theory in Cartesian and Plane Polar Coordinates

In classical orbital theory:

$$\underline{F} = m\dot{\underline{v}} = -\frac{mMG}{r^2}\underline{e}_r \quad -(1)$$

so in Cartesian coordinates:

$$\frac{d^2X}{dt^2}\underline{i} + \frac{d^2Y}{dt^2}\underline{j} = -\frac{mG}{(X^2+Y^2)^{3/2}}(X\underline{i}+Y\underline{j})$$

$$= -mG\frac{\underline{r}}{r^3} \quad -(2)$$

where

$$\underline{r} = X\underline{i} + Y\underline{j} = r\underline{e}_r \quad -(3)$$

It follows that:

$$\ddot{X} = -\frac{mG}{(X^2+Y^2)^{3/2}}X \quad -(4)$$

$$\ddot{Y} = -\frac{mG}{(X^2+Y^2)^{3/2}}Y \quad -(5)$$

Eqns (4) and (5) can be integrated to give the ellipse:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \quad -(6)$$

where a and b are constants. Eqs (4) and (5) can be transformed into the plane polar coordinate system using:

$$X = r\cos\phi \quad -(7)$$
$$Y = r\sin\phi \quad -(8)$$

and

$$\underline{e}_r = \underline{i}\cos\phi + \underline{j}\sin\phi \quad -(9)$$

It follows that:

$$\dot{x} = \frac{d}{dt}\left(r\cos\phi(t)\right) = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \quad -(10)$$
$$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \quad -(11)$$

Furthermore:

$$\ddot{x} = \ddot{r}\cos\phi - \dot{\phi}\sin\phi\,\dot{r} - (\dot{r}\dot{\phi} + r\ddot{\phi})\sin\phi - r\dot{\phi}^2\cos\phi$$
$$= (\ddot{r} - r\dot{\phi}^2)\cos\phi - (2\dot{r}\dot{\phi} + r\ddot{\phi})\sin\phi \quad -(11a)$$

Similarly:

$$\ddot{y} = (\ddot{r} - r\dot{\phi}^2)\sin\phi + (2\dot{r}\dot{\phi} + r\ddot{\phi})\cos\phi. \quad -(12)$$

It follows that:

$$-(13)$$
$$(\ddot{r} - r\dot{\phi}^2)\cos\phi - (2\dot{r}\dot{\phi} + r\ddot{\phi})\sin\phi = -\frac{mG}{r^2}\cos\phi$$

and

$$(\ddot{r} - r\dot{\phi}^2)\sin\phi + (2\dot{r}\dot{\phi} + r\ddot{\phi})\cos\phi = -\frac{mG}{r^2}\sin\phi \quad -(14)$$

It follows that a possible solution of eqs. (13) and (14) is:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{mG}{r^2} \quad -(15)$$

and

$$2\dot{r}\dot{\phi} + r\ddot{\phi} = 0 \quad -(16)$$

Q.E.D.

Eqs. (15) and (16) are the well known equations of the ellipse in plane polar coordinates:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(17)$$

Eqs. (6) and (17) are equivalent representations of the ellipse.

More accurately, the ellipse in eq. (6) must be offset so that its focus is equivalent to eq. (17), where the mass m orbits a mass M situated at one focus of the ellipse. The offset ellipse is:

$$\frac{(X-X_0)^2}{a^2} + \frac{(Y-Y_0)^2}{b^2} = 1 \quad -(18)$$

It should be noted that the centrifugal and Coriolis terms in eqs. (15) and (16) are the result of expressing X and Y by eqs. (7) and (8). They are inherent in the Cartesian eqns. (4) and (5).

It follows that a variety of problems can be solved numerically with the Cartesian coordinates.