375(10) : Lagrangian for a Binary Pulsar

The non-relativistic Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1 + \frac{1}{2} m_2 \dot{\underline{r}}_2 \cdot \dot{\underline{r}}_2 + \frac{m_1 m_2 G}{r} \quad -(1)$$

where

$$r = | \underline{r}_1 - \underline{r}_2 | \quad -(2)$$

This can be expressed as:

$$\mathcal{L} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{\underline{r}} \cdot \dot{\underline{r}} + m_1 m_2 G \frac{|\underline{r}|}{r^2} \quad -(3)$$

where

$$|\underline{r}| = (\underline{r} \cdot \underline{r})^{1/2} \quad -(4)$$

The Euler Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} = \left(\frac{m_1 m_2}{m_1 + m_2} \right) \ddot{\underline{r}} \quad -(5)$$

$$= m_1 m_2 G \frac{\partial}{\partial \underline{r}} \left(\frac{|\underline{r}|}{r^2} \right)$$

$$= m_1 m_2 G \frac{\partial}{\partial \underline{r}} \left(\frac{1}{|\underline{r}|} \right)$$

where:

$$\frac{\partial}{\partial \underline{r}} \frac{1}{(\underline{r} \cdot \underline{r})^{1/2}} = - \frac{1}{2} 2 \underline{r} \frac{1}{(\underline{r} \cdot \underline{r})^{3/2}} \quad -(6)$$

$$= - \frac{\underline{r}}{r^3}$$

Therefore:

2) $\left(\frac{m_1 m_2}{m_1 + m_2}\right) \ddot{\underline{r}} = -m_1 m_2 G \frac{\underline{r}}{r^3}$ — (7)

i.e. $$\boxed{\ddot{\underline{r}} = -(m_1 + m_2) G \frac{\underline{r}}{r^3}} \quad \text{— (8)}$$

If $m_2 \gg m_1$ — (9)

as in the solar system, then:

$$\ddot{\underline{r}} \sim -m_2 G \frac{\underline{r}}{r^3} \quad \text{— (10)}$$

Eq. (8) can be solved numerically by Maxima in any coordinate system.

The force equations are

$$\underline{F}_1 = m_1 \ddot{\underline{r}}_1 \quad \text{— (11)}$$

and $$\underline{F}_2 = m_2 \ddot{\underline{r}}_2 \quad \text{— (12)}$$

where $$\underline{r}_1 = \frac{m_2}{m_1 + m_2} \underline{r} \quad \text{— (13)}$$

$$\underline{r}_2 = -\frac{m_1}{m_1 + m_2} \underline{r} \quad \text{— (14)}$$

and $$m_1 \underline{r}_1 + m_2 \underline{r}_2 = \underline{0} \quad \text{— (15)}$$

From eqs. (8) and (13):

3)

$$\ddot{\underline{r}}_1 = -m_2 G \frac{\underline{r}}{r^3} \qquad -(16)$$

$$= -\frac{m_2 G}{r^3}(\underline{r}_1 - \underline{r}_2)$$

Fm eqs. (8) and (14):

$$\ddot{\underline{r}}_2 = m_1 G \frac{\underline{r}}{r^3} = \frac{m_1 G}{r^3}(\underline{r}_1 - \underline{r}_2) - (17)$$

The three equation of motion are therefore eqs. (8), (16) and (17).

Fm eqs. (15) and (16):

$$\ddot{\underline{r}}_1 = -\frac{m_2 G}{r^3}\underline{r}_1 - \frac{m_1 G}{r^3}\underline{r}_1$$

$$= -(m_1 + m_2) G \frac{\underline{r}_1}{r^3} \qquad -(17)$$

Similarly:

$$\ddot{\underline{r}}_2 = -(m_2 + m_1) G \frac{\underline{r}_2}{r^3} \qquad -(18)$$

The three equations of motion therefore reduce to

$$\boxed{\begin{aligned} \ddot{\underline{r}} &= -(m_1 + m_2) G \, \underline{r} / r^3 \quad -(19) \\ \ddot{\underline{r}}_1 &= -(m_1 + m_2) G \, \underline{r}_1 / r^3 \quad -(20) \\ \ddot{\underline{r}}_2 &= -(m_1 + m_2) G \, \underline{r}_2 / r^3 \quad -(21) \end{aligned}}$$

4) Subtracting eq. (21) from eq. (20) gives eq. (19), Q.E.D. Here:

$$\frac{1}{r^3} = \frac{1}{|\underline{r}_1 - \underline{r}_2|^3} \quad -(22)$$

1) Eq. (19) gives a Newtonian ellipse with mass

$$M = m_1 + m_2 \quad -(23)$$

2) Eqs. (20) and (21) must be solved simultaneously with computer algebra. They are:

$$\underline{\ddot{r}}_1 = -MG \frac{\underline{r}_1}{|\underline{r}_1 - \underline{r}_2|^3} \quad -(24)$$

$$\underline{\ddot{r}}_2 = -MG \frac{\underline{r}_2}{|\underline{r}_1 - \underline{r}_2|^3} \quad -(25)$$

so they are simultaneous differential equations.

In Cartesian coordinates:

$$\underline{r} = X \underline{i} + Y \underline{j} \quad -(26)$$

$$\underline{r}_1 = X_1 \underline{i} + Y_1 \underline{j} \quad -(27)$$

$$\underline{r}_2 = X_2 \underline{i} + Y_2 \underline{j} \quad -(28)$$

and

$$r^3 = (X^2 + Y^2)^{3/2} \quad -(29)$$

By definition:

$$\underline{r} = \underline{r}_1 - \underline{r}_2 \quad -(30)$$

so:

5)

$$X = X_1 - X_2 \quad -(31)$$

$$Y = Y_1 - Y_2 \quad -(32)$$

So

$$\frac{1}{r^3} = \frac{1}{\left((X_1 - X_2)^2 + (Y_1 - Y_2)^2\right)^{3/2}} \quad -(33)$$

These equations define the two body problem in classical non-relativistic gravitation. The general n body problem is very intricate.

Relativistic Lagrangian

This is:

$$\mathcal{L} = -mc^2\left(\left(1 - \frac{\dot{\underline{r}}_1 \cdot \dot{\underline{r}}_1}{c^2}\right)^{1/2} + \left(1 - \frac{\dot{\underline{r}}_2 \cdot \dot{\underline{r}}_2}{c^2}\right)^{1/2}\right) + \frac{m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|} \quad -(34)$$

where

$$r = |\underline{r}_1 - \underline{r}_2| \quad -(35)$$

This is the relativistic two body problem. The centre of mass is still defined by:

$$m_1 \underline{r}_1 + m_2 \underline{r}_2 = \underline{0} \quad -(36)$$

and

$$\underline{r} = \underline{r}_1 - \underline{r}_2 \quad -(37)$$

Therefore

$$\underline{r}_1 = \frac{m_2}{m_1 + m_2}\underline{r} \quad -(38)$$

6) and
$$\underline{r_2} = -\frac{m_1}{m_1+m_2}\,\underline{r} \quad -(39)$$

so the Lagrangian (34) reduces to:

$$\mathcal{L} = -mc^2\left(\left(1-\left(\frac{m_2}{m_1+m_2}\right)^2\frac{\underline{r}\cdot\underline{r}}{c^2}\right)^{1/2} + \left(1-\left(\frac{m_1}{(m_1+m_2)}\right)^2\frac{\underline{r}\cdot\underline{r}}{c^2}\right)^{1/2}\right)$$
$$+\, m_1 m_2 G\,\frac{|\underline{r}|}{r^3} \quad -(40)$$

with Euler Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial\underline{r}} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\underline{r}}} \quad -(41)$$

Eqs. (40) and (41) describe the relativistic orbit of m_1 around m_2. This is a precessing ellipse.

In the Hulse Taylor binary pulsar:

$$m_1 = 2.824\times10^{30}\ \text{kg} \quad -(42)$$
$$m_2 = 2.804\times10^{30}\ \text{kg} \quad -(43)$$

The peri~~apsis~~astron, or distance of closest approach of m_1 to m_2 can be used as the initial condition.

This is:
$$r_{min} = 1.1\ \text{solar radii}$$
$$= 7.6527\times10^{8}\ \text{metres.}$$