B(5): Direct Integration of the ECE2 Hamiltonian

· Consider the Hamiltonian:

$$H_0 = H - mc^2 = (\gamma - 1)mc^2 - \frac{mMG}{r} \quad -(1)$$

where

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(2)$$

and

$$v_N^2 = \dot{r}^2 + r^2\dot{\phi}^2 \quad -(3)$$

plane polar coordinates (r, ϕ). Here v_N is the Newtonian velocity. It follows from eq. (1) that:

$$\gamma - 1 = \frac{1}{mc^2}\left(H_0 + \frac{mMG}{r}\right) \quad -(4)$$

$$-(5)$$

and

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{mc^2}\left(H_0 + \frac{mMG}{r}\right)$$

and

$$1 - \frac{v_N^2}{c^2} = \left(1 + \frac{1}{mc^2}\left(H_0 + \frac{mMG}{r}\right)\right)^{-2} \quad -(6)$$

So

$$\frac{v_N^2}{c^2} = 1 - \frac{1}{\left(1 + \frac{1}{mc^2}\left(H_0 + \frac{mMG}{r}\right)\right)^2} \quad -(7)$$

Now use the binomial expansion:

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \quad -(8)$$

to find that:

2)

$$\frac{v_N^2}{c^2} = 1 - \left(2\left(1+\frac{1}{mc^2}\left(H_0+\frac{mMG}{r}\right)\right) + 3\left(\frac{1}{mc^2}\left(H_0+\frac{mMG}{r}\right)\right)^2 - \dots\right)$$

$$= \frac{2}{mc^2}\left(H_0+\frac{mMG}{r}\right) - \frac{3}{m^2c^4}\left(H_0+\frac{mMG}{r}\right)^2 + \dots \quad -(9)$$

The Newtonian Hamiltonian is:

$$\frac{v_N^2}{c^2} = \frac{2}{mc^2}\left(H_0+\frac{mMG}{r}\right) \quad -(10)$$

i.e. $$H_0 = \frac{1}{2}mv_N^2 - \frac{mMG}{r} \quad -(11)$$

So there is an additional second order term in eq. (9).

Now use:

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} \quad -(12)$$

and the Newtonian result of Lagrangian analysis:

$$\frac{d\phi}{dt} = \frac{L}{mr^2} \quad -(13)$$

where L is a constant angular momentum. So:

$$v_N^2 = \left(\frac{L}{mr^2}\right)^2\left(r^2+\left(\frac{dr}{d\phi}\right)^2\right) \quad -(14)$$

From eqs. (7) and (14):

$$\frac{1}{c^2}\left(r^2+\left(\frac{dr}{d\phi}\right)^2\right)=\left(\frac{mr^2}{L}\right)^2\left(1-\frac{1}{\left(1+\frac{1}{mc^2}\left(H_0+\frac{mMG}{r}\right)\right)^2}\right) \quad -(15)$$

So:

$$\left(\frac{dr}{d\phi}\right)^2=c^2\left(\frac{mr^2}{L}\right)^2\left(1-\frac{1}{\left(1+\frac{1}{mc^2}\left(H_0+\frac{mMG}{r}\right)\right)^2}\right)-r^2 \quad -(16)$$

This quantity can be compared graphically with the Newtonian result from eqs. (11) and (14):

$$v_N^2=\frac{2}{m}\left(H_0+\frac{mMG}{r}\right) \quad -(17)$$

$$=\left(\frac{L}{mr^2}\right)^2\left(r^2+\left(\frac{dr}{d\phi}\right)^2\right)$$

So the Newtonian result is:

$$\left(\frac{dr}{d\phi}\right)^2=\frac{2}{m}\left(\frac{mr^2}{L}\right)^2\left(H_0+\frac{mMG}{r}\right)-r^2 \quad -(18)$$

It is known that the Newtonian orbit is the conic equation:

$$r=\frac{\alpha}{1+\epsilon\cos\phi}. \quad -(19)$$

From eq. (19)

$$\left(\frac{dr}{d\phi}\right)^2=\frac{\epsilon^2 r^4}{\alpha^2}\sin^2\phi \quad -(20)$$

Therefore:

4)
$$\frac{2}{m}\left(\frac{mr^2}{L}\right)^2\left(H_0+\frac{mMG}{r}\right)-r^2=\frac{\epsilon^2 r^4}{\alpha^2}\sin^2\phi \quad -(21)$$

Eq. (21) represents a closed orbit, because the same result is obtained if
$$\phi \rightarrow \phi + 2\pi \quad -(22)$$

If
$$\phi = 2\pi \quad -(23)$$

then
$$r^2=\frac{2}{m}\left(\frac{mr^2}{L}\right)^2\left(H_0+\frac{mMG}{r}\right) \quad -(24)$$

From eqs. (9) and (17) the precessing orbit is given by:
$$v_N^2=\frac{2}{m}\left(H_0+\frac{mMG}{r}\right)-\frac{3}{m^2c^2}\left(H_0+\frac{mMG}{r}\right)^2$$
$$=\left(\frac{L}{mr^2}\right)^2\left(r^2+\left(\frac{dr}{d\phi}\right)^2\right) \quad -(25)$$

i.e.
$$\left(\frac{dr}{d\phi}\right)^2=\frac{2}{m}\left(\frac{mr^2}{L}\right)^2\left(H_0+\frac{mMG}{r}\right)-r^2-\frac{3}{m^2c^2}\left(H_0+\frac{mMG}{r}\right)^2$$
$$=\frac{\epsilon^2 r^4}{\alpha^2}\sin^2\phi-\frac{3}{m^2c^2}\left(H_0+\frac{mMG}{r}\right)^2 \quad -(26)$$

Graphics

Compare eqs. (16) and (20) using:
$$\sin^2\phi = 1-\cos^2\phi = 1-\frac{1}{\epsilon^2}\left(\frac{\alpha}{r}-1\right)^2 \quad -(27)$$