370(8): Extension of the Dumbbell Model in Notes and Protocol by Horst Eckardt.

We have:

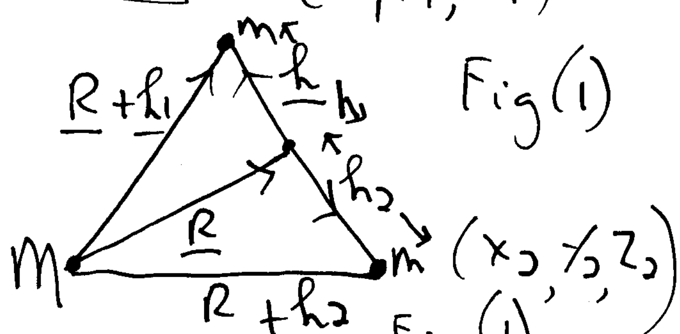

Fig (1)

$$X_1 = |\underline{R} + \underline{h}_1| \sin\theta_1 \cos\phi_1 \quad -(1)$$
$$Y_1 = |\underline{R} + \underline{h}_1| \sin\theta_1 \sin\phi_1 \quad -(2)$$
$$Z_1 = |\underline{R} + \underline{h}_1| \cos\theta_1 \quad -(3)$$

and

$$X_2 = |\underline{R} + \underline{h}_2| \sin\theta_2 \cos\phi_2 \quad -(4)$$
$$Y_2 = |\underline{R} + \underline{h}_2| \sin\theta_2 \sin\phi_1 \quad -(5)$$
$$Z_3 = |\underline{R} + \underline{h}_2| \cos\theta_2 \quad -(6)$$

in spherical polar coordinates. -(7)

The kinetic energy is:

$$T = \frac{1}{2} m \left(\left(\dot{X}_1^2 + \dot{Y}_1^2 + \dot{Z}_1^2 \right) + \left(\dot{X}_2^2 + \dot{Y}_2^2 + \dot{Z}_2^2 \right) \right)$$

$$= \frac{1}{2} m \left(\left(\dot{r}_1^2 + r_1^2 \dot{\theta}_1^2 + r_1^2 \dot{\phi}_1^2 \sin^2\theta_1 \right) + \left(\dot{r}_2^2 + r_2^2 \dot{\theta}_2^2 + r_2^2 \dot{\phi}_2^2 \sin^2\theta_2 \right) \right) \quad -(8)$$

If the two masses m are gravitationally attracted to the mass M then the potential energy is

$$U = -\frac{mMG}{r_1} - \frac{mMG}{r_2} \quad -(9)$$

Here :
$$r_1 = |\underline{R} + \underline{h}_1| \quad -(10)$$
$$r_2 = |\underline{R} + \underline{h}_2| \quad -(11)$$

The Lagrangian is :
$$\mathcal{L} = T - U \quad -(12)$$
The Lagrangian variables are $r_1, r_2, \theta_1, \phi_1, \theta_2$ and ϕ_2. So there are six simultaneous equations defined by :
$$\frac{\partial \mathcal{L}}{\partial r_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_1} \quad -(13)$$
$$\frac{\partial \mathcal{L}}{\partial r_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_2} \quad -(14)$$
$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \quad -(15)$$
$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} \quad -(16)$$
$$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} \quad -(17)$$
$$\frac{\partial \mathcal{L}}{\partial \phi_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} \quad -(18)$$

These equations can be solved for $r_1(t)$,

$r_2(t)$, $\theta_1(t)$, $\phi_1(t)$, $\theta_2(t)$, $\phi_2(t)$, $\dot{r}_1(t)$, $\dot{r}_2(t)$, $\dot{\theta}_1(t)$, $\dot{\phi}_1(t)$, $\dot{\theta}_2(t)$ and $\dot{\phi}_2(t)$

This definition of the problem and its solution takes place in the laboratory frame (X, Y, Z). The line joining the two masses can be thought of as a diameter of the earth.

Check on Concepts

If only one mass is being considered, the Lagrangian reduces to:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) \right) - \frac{mMG}{r} \quad -(19)$$

and the Lagrange variables are r, θ and ϕ. When there are two masses the Lagrange variables become r_1, θ_1, ϕ_1, r_2, θ_2, ϕ_2.

Having found the motion of r_1 and r_2, the motion of R, L_1 and L_2 can also be found from:

$$\underline{r}_1 = \underline{R} + \underline{L}_1 \quad -(20)$$

$$\underline{r}_2 = \underline{R} + \underline{L}_2 \quad -(21)$$
