276(3): Three Dimensional Orbit Theory in Special Relativity

The infinitesimal line element is:

$$ds^2 = c^2 d\tau^2 = (c^2 - v^2)\, dt^2 \quad -(1)$$

and the hamiltonian is:

$$H = (\gamma - 1) m c^2 + U(r) \quad -(2)$$

for any potential of attraction $U(r)$. Sommerfeld showed in 1913 that the hamiltonian (2) produces a precessing ellipse. It is now known that three dimensional orbit theory produces:

$$\boxed{x = \frac{L}{L_Z} = 1 + \frac{3MG}{\alpha c^2}} \quad -(3)$$

for small x as observed experimentally. Therefore 3D orbit theory reduces to x theory for small x. It can produce experimental effects in this limit such as precession of the perihelion, light deflection due to gravitation, the gravitational red shift and so on, as in previous papers on x theory. The origin of the Thomas precession is also eq. (3).

The 3D relativistic theory can be developed with the 3D Lorentz factor:

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(4)$$

where:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\beta}{dt}\right)^2 \quad -(5)$$

The 3D relativistic kinetic energy is:

$$T = (\gamma - 1)mc^2 \quad -(6)$$

If:

$$v \ll c \quad -(7)$$

then:

$$T \rightarrow \frac{1}{2}mv^2. \quad -(8)$$

Now use:

$$\frac{dr}{dt} = \frac{d\beta}{dt}\frac{dr}{d\beta} \quad -(9)$$

to find that:

Classical Limit

$$v^2 = \left(\frac{d\beta}{dt}\right)^2\left(r^2 + \left(\frac{dr}{d\beta}\right)^2\right). \quad -(10)$$

From the Lagrangian analysis of previous papers:

$$\frac{d\beta}{dt} = \frac{L}{mr^2}, \quad -(11)$$

$$\mathcal{L} = \frac{1}{2}mv^2 + U(r) \quad -(11a)$$

where L is the total ~~angular~~ momentum. So:

$$v^2 = \frac{L^2}{m^2 r^4}\left(r^2 + \left(\frac{dr}{d\beta}\right)^2\right) \quad -(12)$$

For an inverse square law of attraction:

$$F(r) = -\frac{mMG}{r^2} \quad -(13)$$

and

$$U(r) = -\frac{mMG}{r} \quad -(14)$$

so

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(15)$$

It follows that:

$$\frac{dr}{d\beta} = -\frac{\epsilon r^2}{\alpha}\sin\beta \quad -(16)$$

and

$$v^2 = \frac{L^2}{m^2 r^4}\left(r^2 + \frac{\epsilon^2 r^4}{\alpha^2}\sin^2\beta\right) \quad -(17)$$

$$= \frac{L^2}{m^2}\left(\frac{1}{r^2} + \frac{\epsilon^2}{\alpha^2}\sin^2\beta\right)$$

Now use:

$$\sin^2\beta = 1 - \cos^2\beta \quad -(18)$$

and:

4)

$$\frac{1}{r^2} = \frac{1}{\alpha^2}\left(1+\epsilon\cos\beta\right)^2 \quad -(19)$$

so:

$$v^2 = \frac{L^2}{m^2\alpha^2}\left(\left(1+\epsilon\cos\beta\right)^2 + \epsilon^2\left(1-\cos^2\beta\right)\right) \quad -(20)$$

so

$$\boxed{v^2 = \frac{L^2}{m^2\alpha^2}\left(1+\epsilon^2+2\epsilon\cos\beta\right)} \quad -(21)$$

The relativistic kinetic energy is:

$$T = \left(\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right)mc^2 \quad -(22)$$

where: $-(23)$

$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L}{L_Z}\right)^2\sin^2\phi\right)^{1/2}}$$

$$= \left(1 - \left(\frac{1}{1-\left(\frac{L_Z}{L}\right)^2}\right)\cos^2\theta\right)^{1/2}$$

Therefore the kinetic energy can be graphed

5) as a function of ϕ, of θ, and of ϕ and θ.

In general:

$$T = T(\phi, \theta). \quad -(24)$$

Relativistic Theory:

$$H = (\gamma - 1) mc^2 + U(r) \quad -(25)$$

$$\mathcal{L} = (\gamma - 1) mc^2 - U(r) \quad -(26)$$

and the four Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(27)$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\beta}} \quad -(28)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(29)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(\cancel{29}\ 30)$$

These equations will produce different relations between β, θ and ϕ and a different force law. However, the complexity of this procedure

) can be bypassed for small x by using eq. (3)

The velocity v appearing in eq. (1) is the non relativistic velocity given by Eq. (21)

In previous papers it was shown that the observed precession:

$$x = 1 + \frac{3MG}{\alpha c^2} \quad -(31)$$

can be attributed to the Thomas precession, which can be calculated from eq. (1). The origin of a small precession was shown from the previous note to the 3D orbit theory, giving:

$$x = \frac{L}{L_Z} \quad -(32)$$

So eqns. (1) to (3) are self consistent.

This must now be demonstrated precisely, because the Thomas precession is the result of rotating the metric (1), and eq. (32) is the result of a classical theory. It is also known from the Sommerfeld theory of the atom that eq. (2) produces a precessing ellipse. Eq. (3) must therefore be the classical limit of a relativistic theory.