

238(12): Animation of \odot Elliptical Orbit.

In the non-relativistic case:

$$\frac{d\theta}{dt} = \frac{L_0}{mr^2} \quad - (1)$$

So

$$dt = \frac{mr^2}{L_0} d\theta \quad - (2)$$

$$t = \int dt = \frac{md^2}{L_0} \int \frac{d\theta}{(1 + \epsilon \cos \theta)^2} \quad - (3)$$

$$= \frac{md^2}{L_0(1-\epsilon^2)} \left[\frac{2}{(1-\epsilon^2)^{1/2}} \tan^{-1} \left(\frac{(1-\epsilon) \tan(\theta/2)}{(1-\epsilon^2)^{1/2}} \right) - \frac{\epsilon \sin \theta}{1 + \epsilon \cos \theta} \right]$$

The trajectory as a function of θ is:

$$\underline{r}(\theta) = r(\theta) (\cos \theta \underline{i} + \sin \theta \underline{j}) \quad - (4)$$

where

$$r(\theta) = \frac{d}{1 + \epsilon \cos \theta} \quad - (5)$$

In this case eq. (3) is too complicated to invert, i.e. θ cannot be found analytically in terms of t . However, it is possible to construct an analytical animation as follows

2)
1) Firstly use:

$$X = \frac{d \cos \theta}{1 + e \cos \theta} \quad - (6)$$

$$Y = \frac{d \sin \theta}{1 + e \cos \theta} \quad - (7)$$

2) Secondly plot (X, Y) as a function of increasing θ , and animate.

3) For each θ calculate t from eq. (3), plot (X, Y) as a function of t and animate.

For example if:

$$(X_1, Y_1) = \left(\frac{d \cos \theta_1}{1 + e \cos \theta_1}, \frac{d \sin \theta_1}{1 + e \cos \theta_1} \right) \quad - (8)$$

this point (X_1, Y_1) occurs at the time:

$$t_1 = \frac{m d^2}{L_0} \int \frac{d\theta_1}{(1 + e \cos \theta_1)^2} \quad - (9)$$

$$= \frac{m d^2}{L_0 (1 - e^2)} \left[\frac{2}{(1 - e^2)^{1/2}} \tan^{-1} \left(\frac{(1 - e) \tan(\theta_1/2)}{(1 - e^2)^{1/2}} \right) - \frac{e \sin \theta_1}{1 + e \cos \theta_1} \right]$$

3) So (X_1, Y_1) is plotted at t_1 .

Now repeat so that:

(X_2, Y_2) is plotted at t_2

and (X_n, Y_n) is plotted at t_n .

This animation produces a particle of mass m moving around the ellipse at the correct speed.

Precessing Ellipse

In this case:

$$\underline{e}_r = \cos \theta \underline{i} + \sin \theta \underline{j} \quad - (10)$$

and

$$r = \frac{d}{1 + e \cos(x\theta)}, \quad - (11)$$

$$t = \frac{md^2}{L_0} \int \frac{d\theta}{(1 + e \cos(x\theta))^2} \quad - (12)$$

so

$$X = \frac{d \cos \theta}{1 + e \cos(x\theta)} \quad - (13)$$

$$Y = \frac{d \sin \theta}{1 + e \cos(x\theta)} \quad - (14)$$

and plot (X, Y) as a function of t .
