238(12): Animation of the Elliptical Orbit.

In the non-relativistic case:

$$\frac{d\theta}{dt} = \frac{L_0}{mr^2} \quad -(1)$$

so

$$dt = \frac{mr^2}{L_0}\, d\theta \quad -(2)$$

$$t = \int dt = \frac{m\alpha^2}{L_0}\int \frac{d\theta}{(1+\epsilon\cos\theta)^2} \quad -(3)$$

$$= \frac{m\alpha^2}{L_0(1-\epsilon^2)}\left[\frac{2}{(1-\epsilon^2)^{1/2}}\tan^{-1}\left(\frac{(1-\epsilon)\tan(\theta/2)}{(1-\epsilon^2)^{1/2}}\right) - \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\right].$$

The trajectory as a function of θ is:

$$\underline{r}(\theta) = r(\theta)\left(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}\right) \quad -(4)$$

where

$$r(\theta) = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(5)$$

In this case eq. (3) is too complicated to invert, i.e. θ cannot be found analytically in terms of t. However, it is possible to construct an analytical animation as follows.

2)

1) Firstly use:

$$X = \frac{\alpha \cos\theta}{1+\epsilon\cos\theta}, \quad -(6)$$

$$Y = \frac{\alpha \sin\theta}{1+\epsilon\cos\theta} \quad -(7)$$

2) Secondly plot (X, Y) as a function of increasing θ, and animate.

3) For each θ calculate t from eq. (3), plot (X, Y) as a function of t and animate.

For example if:

$$(X_1, Y_1) = \left(\frac{\alpha\cos\theta_1}{1+\epsilon\cos\theta_1}, \frac{\alpha\sin\theta_1}{1+\epsilon\cos\theta_1}\right) \quad -(8)$$

thus point (X_1, Y_1) occurs at the time:

$$t_1 = \frac{m\alpha^2}{L_0}\int\frac{d\theta_1}{(1+\epsilon\cos\theta_1)^2} \quad -(9)$$

$$= \frac{m\alpha^2}{L_0(1-\epsilon^2)}\left[\frac{2}{(1-\epsilon^2)^{1/2}}\tan^{-1}\left(\frac{(1-\epsilon)\tan(\theta_1/2)}{(1-\epsilon^2)^{1/2}}\right) - \frac{\epsilon\sin\theta_1}{1+\epsilon\cos\theta_1}\right]$$

3) So (X_1, Y_1) is plotted at t_1.

Now repeat so that:

(X_2, Y_2) is plotted at t_2

and (X_n, Y_n) is plotted at t_n.

This animation produces a particle of mass m moving around the ellipse at the correct speed.

<u>Processing Ellipse</u>

In this case:

$$\underline{e}_r = \cos\theta \, \underline{i} + \sin\theta \, \underline{j} \quad -(10)$$

and

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)}, \quad -(11)$$

$$t = \frac{m\alpha^2}{L_0}\int \frac{d\theta}{(1+\epsilon\cos(x\theta))^2} \quad -(12)$$

so

$$X = \frac{\alpha\cos\theta}{1+\epsilon\cos(x\theta)} \quad -(13)$$

$$Y = \frac{\alpha\sin\theta}{1+\epsilon\cos(x\theta)} \quad -(14)$$

and plot (X, Y) as a function of t.