Change of Variable in Note 229(5)

Change variables to

$u = \frac{r}{a}$ noting that

$u0 = \frac{R_0}{a}$

so that

$du = \frac{dr}{a}$

In this note, the exponential term

$\int_a^b \text{Sqrt}[V - E]\, dr$

can shown to reduce to

$$\int_1^\gamma \sqrt{-\frac{1}{e^{u-\frac{R_0}{a}}+1} - \lambda + \frac{\beta}{u} + 1}\, du$$

```
u0 = R0/a;
v0 * (1 - 1/(1 + Exp[r/a - R0/a]));
v1 = % /. r/a → u
```

$$v0\left(1 - \frac{1}{e^{u-\frac{R_0}{a}} + 1}\right)$$

The Coulomb potential reduces using

$u = \frac{r}{a}$

```
z1 z2 e^2 / (4 π ϵ0 a u);
% /. a -> R0 / u0
```

$$\frac{e^2 z_1 z_2}{4 \pi a u \epsilon_0}$$

Simplify this expression by writing

$$c_1 = \frac{e^2 x0 z_1 z_2}{4 \pi R_0 \epsilon_0}$$

```
v2 = % /. (e^2 u0 z1 z2)/(4 π R0 ϵ0) -> c1
```

$$\frac{c_1}{u}$$

```
v = v1 + v2;
Expand[(% - e0) / v0]
```

$$-\frac{1}{e^{u-\frac{R_0}{a}}+1}+\frac{c_1}{u\, v0}-\frac{e_0}{v0}+1$$

Substitute

$$\lambda = \frac{e_0}{v0} \quad \text{and} \quad \beta = \frac{c_1}{v0}$$

```
% /. e0/v0 → λ;
% /. c1/v0 → β;
integrand = Sqrt[%]
```

$$\sqrt{-\frac{1}{e^{u-\frac{R_0}{a}}+1}-\lambda+\frac{\beta}{u}+1}$$

- **Solution check**

$$\text{integrand} \;/.\; \lambda \to \frac{e_0}{v0};$$
$$\% \;/.\; \beta \to \frac{c_1}{v0};$$
$$\% \;/.\; c_1 \to \frac{e^2\, u0\, z_1\, z_2}{4\,\pi\, R_0\, \epsilon_0};$$
$$\% \;/.\; u \to r/a$$

$$\sqrt{-\frac{1}{e^{\frac{r}{a}-\frac{R_0}{a}}+1}+\frac{e^2\, z_1\, z_2}{4\,\pi\, r\, v0\, \epsilon_0}-\frac{e_0}{v0}+1}$$

This is the original form of the integrand for note 229 (5)