220(4): Expression for $\theta(t)$ and Fundamental Conserved Quantities.

The fundamental assumption in the three particle problem is that

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3. \quad -(1)$$

The conserved quantities in each lagrangian are the three total energies, E_1, E_2 and E_3, and the three angular momenta L_1, L_2 and ~~E_3~~ L_3:

$$E_1 = -\frac{k_1}{2a_1}, \quad E_2 = -\frac{k_2}{2a_2}, \quad E_3 = -\frac{k_3}{2a_3}, \quad -(2)$$

$$L_1 = \mu_1 R_1^2 \dot{\theta}_1, \quad L_2 = \mu_2 R_2^2 \dot{\theta}_2, \quad L_3 = \mu_3 R_3^2 \dot{\theta}_3 \quad -(3)$$

Therefore a_1, a_2 and a_3 are also conserved, and defined by:

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}\frac{E_2}{E_1}, \quad \frac{a_1}{a_3} = \frac{m_3}{m_2}\frac{E_3}{E_1}, \quad \frac{a_2}{a_3} = \frac{m_1}{m_2}\frac{E_3}{E_2} \quad -(4)$$

It is convenient at this point to summarize the theory of the two particle orbit in order to obtain an expression for $\theta(t)$, then extend this to the three particle problem, then introduce the new universal law of gravitation. See Marion and Thornton, 3rd ed., pp. 261 ff.

The two fundamental observables of any 2 particle orbit are τ and ϵ. Here τ is the time taken for one revolution and ϵ is the eccentricity. The area of the ellipse is πab, where a and b are the semi major and minor axes. So:

$$\frac{\pi ab}{\tau} t = \int dA \quad -(5)$$

where

$$dA = \frac{1}{2} r^2 d\theta. \quad -(6)$$

So Kepler's second law is:

$$\frac{dA}{dt} = \frac{L}{2\mu} = \text{constant}. \quad -(7)$$

If the initial condition is such that:

$$\theta = 0 \text{ at } t = 0 \quad -(8)$$

then

$$\frac{\pi ab}{\tau} t = \frac{1}{2} \int_0^\theta r^2 d\theta \quad -(9)$$

where

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(10)$$

So:

$$\boxed{\frac{\pi ab}{\tau} t = \frac{\alpha^2}{2} \int_0^\theta \frac{d\theta}{(1 + \epsilon \cos\theta)^2}} \quad -(11)$$

The right hand side is a standard integral:

$$\int^\theta \frac{d\theta}{(1+\epsilon\cos\theta)^2} = \frac{1}{1-\epsilon^2} \left[\frac{2}{(1-\epsilon^2)^{1/2}} \tan^{-1}\left(\frac{(1-\epsilon)\tan\frac{\theta}{2}}{(1-\epsilon^2)^{1/2}} \right) - \frac{\epsilon \sin\theta}{1+\epsilon\cos\theta} \right] \quad -(12)$$

Now use:

$$ab = (1-\epsilon^2)^{-3/2} \quad -(13)$$

so

$$\frac{2\pi t}{\tau} = 2\tan^{-1}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/2}\tan\frac{\theta}{2}\right) - \epsilon\frac{(1-\epsilon^2)^{1/2}\sin\theta}{1+\epsilon\cos\theta} \quad -(14)$$

In standard astronomy this equation is inverted:

$$\theta(t) = \frac{2\pi t}{\tau} + 2\epsilon\sin\frac{2\pi t}{\tau} + \frac{5}{4}\epsilon^2\sin\frac{4\pi t}{\tau} + \frac{1}{12}\epsilon^3\left(13\sin\frac{6\pi t}{\tau} - 3\sin\frac{2\pi t}{\tau}\right) + \dots \quad -(15)$$

this is a very cumbersome equation, but it serves the purpose of showing that θ is defined by τ and ϵ, and that θ depends on time.

A more straightforward method is to use the fact that:

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha}\sin\theta \quad -(16)$$

where

$$\sin\theta = \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} = \frac{1}{\epsilon r}\left(r^2\epsilon^2 - (\alpha - r)^2\right)^{1/2} \quad -(17)$$

4)

So:

$$\frac{d\theta}{dr} = \frac{\alpha}{r(r^2\epsilon^2-(\alpha-r)^2)^{1/2}} \quad -(18)$$

i.e.

$$d\theta = \frac{\alpha}{r(r^2\epsilon^2-(\alpha-r)^2)^{1/2}}\, dr \,. \quad -(19)$$

therefore in eq. (9):

$$\frac{\pi ab}{\tau}\, t = \frac{1}{2}\int_0^{\theta} \frac{\alpha r \; dr}{(r^2\epsilon^2-(\alpha-r)^2)^{1/2}} \quad -(20)$$

where

$$\theta = \cos^{-1}\left(\frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right)\right) \quad -(21)$$

This procedure produces t as a function of r:

$$\frac{\pi ab}{\tau}\frac{dt}{dr} = \frac{1}{2}\frac{\alpha r}{(r^2\epsilon^2-(\alpha-r)^2)^{1/2}}\,, \quad -(22)$$

i.e.

$$\frac{dt}{dr} = \left(\frac{\tau\alpha}{2\pi ab}\right)\frac{r}{(r^2\epsilon^2-(\alpha-r)^2)^{1/2}}\,, \quad -(23)$$

or

$$\frac{dr}{dt} = \frac{2\pi ab}{\tau\alpha}\,\frac{\left(r^2\epsilon^2-(\alpha-r)^2\right)^{1/2}}{r}$$

$$\frac{dr}{dt} = \left(\frac{2\pi}{\tau}\right)\frac{\alpha\left(r^2\epsilon^2-(\alpha-r)^2\right)^{1/2}}{r(1-\epsilon^2)^{3/2}} \quad -(24)$$

5) using: $$ab = d^2(1-\epsilon^2)^{-3/2} \quad -(25)$$

$$d = a(1-\epsilon^2) \quad -(26)$$

Therefore:

$$\boxed{\frac{dr}{dt} = \frac{2\pi d}{\tau}\left(\frac{(r^2\epsilon^2-(d-r)^2)^{1/2}}{r(1-\epsilon^2)^{3/2}}\right)} \quad -(27)$$

Now use:

$$\frac{d\theta}{dt} = \frac{d\theta}{dr}\frac{dr}{dt} \quad -(28)$$

$$= \frac{2\pi}{\tau}\left(\frac{d}{r}\right)^2(1-\epsilon^2)^{-3/2}$$

$$\frac{d\theta}{dt} = \frac{2\pi}{\tau}\left(\frac{a}{r}\right)^2(1-\epsilon^2)^{-1/2} \quad -(28)$$

Eq. (28) is: $$\frac{d\theta}{dt} = \frac{2\pi ab}{\tau} \cdot \frac{1}{r^2}$$

$$\boxed{\frac{d\theta}{dt} = \frac{2A}{\tau} \cdot \frac{1}{r^2}} \quad -(29)$$

where
$$A = \pi ab \quad -(30)$$

is the area of the ellipse. However:

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2} \quad -(31)$$

so
$$\boxed{L = \frac{2A}{\tau}\mu} \quad -(32)$$

b) From eq. (27):

$$\int dt = \frac{\tau}{2\pi\alpha}\int \frac{r(1-\epsilon^2)^{3/2}\,dr}{(r^2\epsilon^2-(\alpha-r)^2)^{1/2}} \qquad -(33)$$

$$= \frac{\tau(1-\epsilon^2)^{3/2}}{2\pi\alpha}\int \frac{r\,dr}{(r^2\epsilon^2-(\alpha-r)^2)^{1/2}}$$

and from eq. (11):

$$\frac{\pi ab}{\tau}\,t = \frac{\alpha^2}{2}\int_0^\theta \frac{d\theta}{(1+\epsilon\cos\theta)^2} \qquad -(34)$$

$$t = \frac{\tau\alpha^2}{2\pi ab}\int_0^\theta \frac{d\theta}{(1+\epsilon\cos\theta)^2}$$

$$\boxed{t = \frac{(1-\epsilon^2)^{3/2}\tau}{2\pi}\int_0^\theta \frac{d\theta}{(1+\epsilon\cos\theta)^2}} \qquad -(35)$$

Instead of inverting eq. (35) to try to find $\theta(t)$, it is much more straightforward to measure the time t taken for a given orbit to transcend an angle θ.

Using the new gravitational potential

The time t in eq. (35) will be changed to

$$\boxed{t = \frac{(1-\epsilon^2)^{3/2}\tau}{2\pi}\int_0^\theta \frac{d\theta}{(1+\epsilon\cos(x\theta))^2}}$$

$-(36)$

). In the three particle problem in the Newtonian limit there are three times:

$$t_1 = \frac{(1-\epsilon_1^2)^{3/2}\tau_1}{2\pi}\int_0^{\theta_1}\frac{d\theta_1}{(1+\epsilon_1\cos(x_1\theta_1))^2} \quad -(37)$$

$$t_2 = \frac{(1-\epsilon_2^2)^{1/2}\tau_2}{2\pi}\int_0^{\theta_2}\frac{d\theta_2}{(1+\epsilon_2\cos(x_2\theta_2))^2} \quad -(38)$$

$$t_3 = \frac{(1-\epsilon_3^2)^{1/2}\tau_3}{2\pi}\int_0^{\theta_3}\frac{d\theta_3}{(1+\epsilon_3\cos\theta_3)^2} \quad -(39)$$

Note carefully that it requires a time τ for the radius vector to sweep out the area of the ellipse:

$$A = \pi ab = \frac{\pi\alpha^2}{(1-\epsilon^2)^{3/2}} \quad -(40).$$

By Kepler's second law:

$$\frac{dA}{dt} = \frac{L}{2\mu} = \text{constant} \quad -(41)$$

so

$$dt = \frac{2\mu}{L}\,dA \quad -(42)$$

the area A is swept out in a time τ:

$$\int_0^{\tau} dt = \frac{2\mu}{L}\int_0^{A} dA \quad -(43)$$

so

$$\tau = \frac{2\mu}{L}A \quad -(44)$$

which is eq. (32), QED