1)

197(4): A New General Relativity of Orbits

The basic hypothesis is that a stable orbit is due to a constant Cartan torsion of spacetime. The first Cartan structure equation defines torsion as:

$$T = D \wedge q \quad -(1)$$

in the minimal notation of ECE theory. The concept of torsion is a concept of non-Euclidean geometry based on the idea of spacetime connection. The Riemann torsion is defined by:

$$T^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu} \quad -(2)$$

where $\Gamma^{\kappa}_{\mu\nu}$ is the connection, an idea introduced not by Riemann, but by Christoffel. Using the tetrad postulate, eq. (2) is equivalent to:

$$T^{a} = d \wedge q^{a} + \omega^{a}{}_{b} \wedge q^{b} \quad -(3)$$

where $\wedge$ is the wedge product of Cartan. Here $\omega^{a}{}_{b}$ is the spin connection and q^{b} is the Cartan tetrad. The latter is a matrix defined by:

$$V^{a} = q^{a}{}_{\mu} V^{\mu} \quad -(4)$$

where V^{a} and V^{μ} are vectors.

In his original development, Cartan used the tetrad to relate a vector V^{a} defined in Minkowski spacetime to a vector V^{μ} of the general spacetime. The metric of the general spacetime was then defined as:

$$g_{\mu\nu} = q^{a}{}_{\mu} q^{b}{}_{\nu} \eta_{ab} \quad -(5)$$

where η_{ab} is the metric of Minkowski spacetime.

Consider now the case of rotation in a plane, through the angle θ. In this case:

$$\begin{bmatrix} V_x' \\ V_y' \\ V_z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}. \quad -(6)$$

By applying Cartan's basic idea the rotation tetrad is:

$$q^a_\mu = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad -(7)$$

The torsion is:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu. \quad -(8)$$

The rotation of the vector is equivalent however to the rotation of the frame of reference in the opposite sense, and the rotation of the frame is defined by the connection. Therefore:

$$\boxed{\partial_\mu q^a_\nu - \partial_\nu q^a_\mu = \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu} \quad -(9)$$

The left hand side of eq. (9) represents the rotation of the vector with axes fixed. The right hand side of eq. (9) represents the rotation of the axes with vector fixed. The two processes produce the same result.

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3)

$$\Gamma^{a}_{\mu\nu} = -\Gamma^{a}_{\nu\mu} \quad -(10)$$

where:

$$\Gamma^{a}_{\mu\nu} = \partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu b} q^{b}_{\nu} \quad -(11)$$

Therefore:

$$\partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu b} q^{b}_{\nu} = -\left(\partial_\nu q^{a}_{\mu} + \omega^{a}_{\nu b} q^{b}_{\mu}\right) \quad -(12)$$

i.e. :

$$\partial_\mu q^{a}_{\nu} + \partial_\nu q^{a}_{\mu} + \omega^{a}_{\mu b} q^{b}_{\nu} + \omega^{a}_{\nu b} q^{b}_{\mu} = 0 \quad -(13)$$

while eq. (9) means:

$$\partial_\mu q^{a}_{\nu} - \partial_\nu q^{a}_{\mu} - \omega^{a}_{\mu b} q^{b}_{\nu} + \omega^{a}_{\nu b} q^{b}_{\mu} = 0 \quad -(14)$$

Adding eqs. (13) and (14):

$$\boxed{\partial_\mu q^{a}_{\nu} + \omega^{a}_{\nu b} q^{b}_{\mu} = 0} \quad -(15)$$

Subtracting eq. (14) from eq. (13):

$$\boxed{\partial_\nu q^{a}_{\mu} + \omega^{a}_{\mu b} q^{b}_{\nu} = 0} \quad -(16)$$

in which:

$$\omega^{a}_{\nu\mu} = \omega^{a}_{\nu b} q^{b}_{\mu}, \quad -(17)$$

$$\omega^{a}_{\mu\nu} = \omega^{a}_{\mu b} q^{b}_{\nu}. \quad -(18)$$

So:

$$\partial_\mu q^{a}_{\nu} + \omega^{a}_{\nu\mu} = 0 \quad -(19)$$

$$\partial_\nu q^{a}_{\mu} + \omega^{a}_{\mu\nu} = 0. \quad -(20)$$

4)

Therefore the spin connection for a single rotation may be found from eqs. (19) and (20).

From this simple analysis a number of concepts may be introduced: the infinitesimal rotation generator, the angular momentum, the torque, and the angular energy. The angular energy is purely kinetic, there is no potential energy in this analysis. In the dual line element general relativity the energy was also kinetic energy. A stable orbit is the result of constant spacetime torsion, so the angular energy and lagrangian is:

$$\mathcal{L} = \omega J \quad -(21)$$

where J is the magnitude of the constant spacetime angular momentum and ω is the angular velocity:

$$\omega = \frac{d\theta}{dt} \quad -(22)$$

Therefore:

$$\mathcal{L} = J\frac{d\theta}{dt} = J\frac{d\theta}{dr}\frac{dr}{dt}, \quad -(23)$$

i.e.

$$\boxed{\mathcal{L} = J\dot{\theta}} \quad -(24)$$

and the Euler Lagrange equation is:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta} \quad -(25)$$

$$= 0$$

5) because J is a constant of motion.

The next step in the theory is to develop the concept of angular momentum in terms of the rotation defined by eq. (9). The angular momentum is therefore developed as the rotation of spacetime itself.

For this purpose it is useful to develop a concept from quantum mechanics, the angular momentum operator:

$$\hat{J}_z = \frac{\hbar}{i}\frac{\partial}{\partial\theta} \quad -(26)$$

The tetrad in eq. (7) may be identified as in UFT 110 with the rotation matrix, $R_z(\theta)$:

$$R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(27)$$

and the rotation generator:

$$\hat{j}_z = \frac{1}{i}\left.\frac{dR_z(\theta)}{d\theta}\right|_{\theta=0} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The angular momentum operator in quantum mechanics is then

$$\hat{J}_z = \hbar\,\hat{j}_z \quad -(28)$$

These concepts are generalized as in the next note to show that the angular momentum in general is:

$$\boxed{J^a_{\mu\nu} \propto T^a_{\mu\nu}} \quad -(29)$$