

# **NEW ECE EQUATION FROM THE FUNDAMENTAL DEFINITION OF THE TETRAD AND APPLICATION TO LOW ENERGY NUCLEAR REACTION**

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[www.et3m.net](http://www.et3m.net))*

New classical and quantum equations of motion are derived from the fundamental definition of the tetrad in differential geometry and reduced to the Schrödinger equation in a well defined approximation. This is a rigorously self consistent ECE theory that produces more information in general than the Schrödinger equation. The latter is applied to low energy nuclear reaction (LENR) to give a straightforward explanation from differential geometry. It is concluded that LENR is due to a combination of quantum tunnelling and resonant absorption of spacetime quanta such as photons, phonons or plasmons.

*Keywords:* ECE theory, derivation of new equations of motion from the tetrad, Schrödinger limit, low energy nuclear reactions.

## **1. INTRODUCTION**

The ECE unified field theory [1 - 10] is based on differential geometry, and for this reason provides a relatively simple unification of physics. It reduces to all the well known equation of physics, and as in this paper, gives new equations of motion containing novel information, and which reduce in well defined limits to their known counterparts. In Section 2, a new fundamental definition of the generalized Cartan tetrad is given, and developed into classical and quantum equations of motion. This method defines the fundamental Schrödinger postulate on the ECE level, and also defines the minimal prescription rigorously within the framework of ECE theory. The new quantized equation of motion is reduced straightforwardly to the Schrödinger equation using a well defined ap-

proximation, which however loses a great deal of information inherent in differential geometry itself. In Section 3 the Schrödinger equation thus derived is applied to give a straightforward explanation of low energy nuclear reaction (LENR) in terms of quantum tunnelling and simultaneous resonant absorption of quanta of spacetime energy such as photons, phonons or plasmons.

## 2. Derivation of the new tetrad definition and equations of motion

Consider the vector field  $V$  in any dimension. The Cartan tetrad [11] was defined originally by:

$$V^a = q_\mu^a V^\mu \quad (1)$$

where  $a$  indicates the tangent spacetime at a point  $P$  to the base manifold labelled  $\mu$ . The tetrad is a square matrix. During the course of development of ECE theory [1 - 10] this definition of the tetrad has been extended to a vector field  $V$  defined by two different representations in one mathematical space. For example  $(a)$  can denote the complex circular basis and  $\mu$  the Cartesian basis. The tetrad can be used [11] with the components and basis vectors of the complete vector field  $V$ . For example:

$$\begin{bmatrix} e^{(0)} \\ \mathbf{e}^{(1)} \\ \mathbf{e}^{(2)} \\ \mathbf{e}^{(3)} \end{bmatrix} = \begin{bmatrix} q_0^{(0)} & q_1^{(0)} & q_2^{(0)} & q_3^{(0)} \\ q_0^{(1)} & q_1^{(1)} & q_2^{(1)} & q_3^{(1)} \\ q_0^{(2)} & q_1^{(2)} & q_2^{(2)} & q_3^{(2)} \\ q_0^{(3)} & q_1^{(3)} & q_2^{(3)} & q_3^{(3)} \end{bmatrix} \begin{bmatrix} e^0 \\ \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{bmatrix} \quad (2)$$

where  $e^{(a)}$  denote the four unit vectors in a four dimensional spacetime whose space components are represented by the complex circular basis [1–10], and where  $\mathbf{e}^\mu$  denote their counterparts using a Cartesian basis. The complex circular basis is defined by:

$$\mathbf{e}^{(1)} = \frac{1}{\sqrt{2}} (\mathbf{i} - i\mathbf{j}) \quad (3)$$

$$\mathbf{e}^{(2)} = \frac{1}{\sqrt{2}} (\mathbf{i} + i\mathbf{j}) \quad (4)$$

$$\mathbf{e}^{(3)} = \mathbf{k} \quad (5)$$

with cyclically symmetric relations:

$$\mathbf{e}^{(1)} \times \mathbf{e}^{(2)} = i\mathbf{e}^{(3)*} \quad (6)$$

et cyclicum

and the Cartesian basis is defined by

$$\begin{aligned} \mathbf{e}^1 &= \mathbf{i}, \mathbf{e}^2 = \mathbf{j}, \\ \mathbf{e}^3 &= \mathbf{k} \end{aligned} \quad (7)$$

with cyclically symmetric relations:

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ &\text{et cyclicum} \end{aligned} \quad (8)$$

For simplicity of argument consider the transverse components only:

$$\begin{bmatrix} \mathbf{e}^{(1)} \\ \mathbf{e}^{(2)} \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \end{bmatrix} \quad (9)$$

and multiply both sides of Eq. (9) [by  $\mathbf{i} \mathbf{j}$ ]. Then:

$$\begin{bmatrix} \mathbf{e}^{(1)} \\ \mathbf{e}^{(2)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{i} \mathbf{j} \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{i} \mathbf{j} \end{bmatrix} \quad (10)$$

The left hand side is:

$$\begin{bmatrix} \mathbf{e}^{(1)} \cdot \mathbf{i}^{(1)} & \mathbf{e}^{(1)} \cdot \mathbf{j} \\ \mathbf{e}^{(2)} \cdot \mathbf{i}^{(2)} & \mathbf{e}^{(2)} \cdot \mathbf{j} \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{i} \cdot \mathbf{i} & \mathbf{i} \cdot \mathbf{j} \\ \mathbf{j} \cdot \mathbf{i} & \mathbf{j} \cdot \mathbf{j} \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} \quad (11)$$

Therefore in general

$$q_{\mu}^a = \mathbf{e}^a \cdot \mathbf{e}^{\mu T} \quad (12)$$

which is a new and useful definition of the tetrad. Here:

$$\mathbf{e}^a = \begin{bmatrix} \mathbf{e}^{(0)} \\ \mathbf{e}^{(1)} \\ \mathbf{e}^{(2)} \\ \mathbf{e}^{(3)} \end{bmatrix}, \mathbf{e}^{\mu T} = [\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3] \quad (13)$$

The transverse tetrad components are given by:

$$q_{\mu}^a = \begin{bmatrix} \mathbf{e}^{(1)} \cdot \mathbf{i} & \mathbf{e}^{(1)} \cdot \mathbf{j} \\ \mathbf{e}^{(2)} \cdot \mathbf{i} & \mathbf{e}^{(2)} \cdot \mathbf{j} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \quad (14)$$

i.e.:

$$\begin{aligned} q_1^{(1)} &= \frac{1}{\sqrt{2}}, & q_2^{(1)} &= \frac{-i}{\sqrt{2}} \\ q_1^{(2)} &= \frac{i}{\sqrt{2}}, & q_2^{(2)} &= \frac{1}{\sqrt{2}} \end{aligned} \quad (15)$$

and the complete transverse tetrad vector is:

$$\mathbf{q}^{(1)} = \frac{1}{\sqrt{2}}(\mathbf{i} - i \mathbf{j}) \quad (16)$$

$$\mathbf{q}^{(2)} = \frac{1}{\sqrt{2}}(\mathbf{i} + i \mathbf{j}) \quad (17)$$

Therefore Eqs. (15) are the components of Eqs. (16) and (17).

In general, for any vectors  $\mathbf{V}^a$  and  $\mathbf{V}^\mu$ , the tetrad is defined by:

$$q_\mu^a := \frac{\mathbf{V}^a \cdot \mathbf{V}^{\mu T}}{\mathbf{V}^\mu \cdot \mathbf{V}_\mu} \quad (18)$$

where summation is implied over repeated  $\mu$  indices.

For the linear momentum field  $p$ :

$$q_\mu^a = \frac{\mathbf{p}^a \cdot \mathbf{p}^{\mu T}}{\mathbf{p}^\mu \cdot \mathbf{p}_\mu} \quad (19)$$

where:

$$p^\mu p_\mu = m^2 c^2 \quad (20)$$

by the Einstein energy equation in the limit of special relativity. In general relativity:

$$p^\mu p_\mu = \hbar^2 R \quad (21)$$

where  $R$  is defined by the ECE wave equation:

$$(\square + R) q_\mu^a = 0 \quad (22)$$

Now define the momentum tetrad (UFT143 on [www.aias.us](http://www.aias.us)):

$$P_\mu^a = p_0 q_\mu^a \quad (23)$$

where  $p_0$  is to be defined. A new classical equation of motion can be deduced:

$$P_\mu^a = \left( \frac{P_0}{m^2 c^2} \right) \mathbf{p}^a \cdot \mathbf{p}^{\mu T} \quad (24)$$

in the limit of special relativity. The quantized version of this equation is:

$$\hat{P}_\mu^a \psi = \left( \frac{P_0}{m^2 c^2} \right) (\hat{\mathbf{p}}^a \cdot \hat{\mathbf{p}}^{\mu T}) \psi \quad (25)$$

where  $\psi$  is the wavefunction. Eq. (25) may also be interpreted as the Schrödinger postulate on the ECE level of theory. The usual Schrödinger postulate is:

$$p^\mu = i\hbar \partial^\mu \quad (26)$$

In the early development of ECE theory [1–10], the electromagnetic potential was defined as:

$$A_\mu^a = A_0 q_\mu^a \quad (27)$$

so it is possible to define a minimal prescription:

$$p_\mu^a \rightarrow p_\mu^a + eA_\mu^a \quad (28)$$

To illustrate the meaning of Eq. (25) consider:

$$\hat{p}_0^{(0)} \Psi = \left( \frac{P_0}{m^2 c^2} \right) \hat{p}^{(0)} \hat{p}_0^T \Psi \quad (29)$$

$$\hat{p}_i^{(i)} \Psi = \left( \frac{P_0}{m^2 c^2} \right) \hat{\mathbf{p}}^{(i)} \cdot \hat{\mathbf{p}}^{(i)T} \Psi \quad (30)$$

$i = 1, 2, 3$

where:

$$\hat{p}^{(0)} \hat{p}^{0T} = \hat{p}^{(0)} \hat{p}^{0T} \quad (31)$$

$$\hat{p}^{(3)} \cdot \hat{p}^{3T} = \hat{p}^{(3)} \cdot \hat{p}^{3T} \quad (32)$$

$$\hat{\mathbf{p}}^{(1)} \hat{\mathbf{p}}^{1T} = \frac{1}{\sqrt{2}} (\mathbf{p}^1 - i\mathbf{p}^2) \cdot \mathbf{p}^{iT} = \frac{1}{\sqrt{2}} \mathbf{p}^1 \cdot \mathbf{p}^{1T} \quad (33)$$

$$\hat{\mathbf{p}}^{(2)} \hat{\mathbf{p}}^{2T} = \frac{1}{\sqrt{2}} (\mathbf{p}^1 - i\mathbf{p}^2) \cdot \mathbf{p}^{2T} = \frac{1}{\sqrt{2}} \mathbf{p}^2 \cdot \mathbf{p}^{2T} \quad (34)$$

It follows that:

$$\begin{aligned} & \left( \hat{p}_0^{(0)} - \sqrt{2} \hat{p}_1^{(1)} + i\sqrt{2} \hat{p}_2^{(2)} - \hat{p}_3^{(3)} \right) \Psi \\ & = \frac{P_0}{m^2 c^2} p^\mu p_\mu \Psi = p_0 \Psi \end{aligned} \quad (35)$$

giving a new linear quantum equation:

$$\left( \hat{p}_0^{(0)} - \sqrt{2} (\hat{p}_1^{(1)} - i\hat{p}_2^{(2)}) - \hat{p}_3^{(3)} \right) \Psi = p_0 \Psi \quad (36)$$

for the momentum. The eigenoperator on the left hand side of this equation gives eigenvalues  $p_0$  from the eigenfunction. In Eq. (35) use:

$$p^\mu p_\mu = -\hbar^2 \square \quad (37)$$

to find that

$$-\left( \frac{\hbar^2 p_0}{m^2 c^2} \right) \square \Psi = \left( \hat{p}_0^{(0)} - \sqrt{2} (\hat{p}_1^{(1)} - i\hat{p}_2^{(2)}) - \hat{p}_3^{(3)} \right) \Psi \quad (37a)$$

so the d'Alembertian operator may be defined by:

$$\square = -\left( \frac{m^2 c^2}{\hbar^2 p_0} \right) \left( \hat{p}_0^{(0)} - \sqrt{2} (\hat{p}_1^{(1)} - i\hat{p}_2^{(2)}) - \hat{p}_3^{(3)} \right) \quad (38)$$

i.e. in terms of a combination of component operators.

In the special case of movement along the  $Z$  axis:

$$\square \psi = - \left( \frac{m^2 c^2}{\hbar^2 p_0} \right) \hat{p}_0^{(0)} - \hat{p}_3^{(3)} \quad (39)$$

which implies that:

$$\left( \hat{p}_0^{(0)} - \hat{p}_3^{(3)} \right) \psi = p_0 \psi \quad (40)$$

From the general result (19) it follows that:

$$\hat{p}_0^{(0)} \psi = \left( \frac{p_0}{m^2 c^2} \right) \left( \hat{p}^{(0)} \hat{p}^{0T} \right) \psi \quad (41)$$

and

$$\hat{p}_3^{(3)} \psi = \left( \frac{p_0}{m^2 c^2} \right) \left( \hat{\mathbf{p}}^{(3)} \cdot \hat{\mathbf{p}}^{3T} \right) \psi \quad (42)$$

where:

$$\hat{p}^{(0)} \hat{p}^{0T} = \hat{p}^{(0)} \hat{p}_0 = \hat{p}^0 \hat{p}_0 \quad (43)$$

and:

$$\hat{\mathbf{p}}^{(3)} \cdot \hat{\mathbf{p}}^{3T} = -p^3 p_3 \quad (44)$$

So

$$-\hbar^2 \square \psi = \left( \hat{p}^0 \hat{p}_0 + \hat{p}^3 \hat{p}_3 \right) \psi \quad (45)$$

However, the Klein Gordon equation is:

$$\left( \square + \left( \frac{mc}{\hbar} \right)^2 \right) \psi = 0 \quad (46)$$

so:

$$\left( \hat{p}^0 \hat{p}_0 + \hat{p}^3 \hat{p}_3 \right) \psi = m^2 c^2 \psi \quad (47)$$

i. e.

$$\left( \hat{p}^{(0)} \hat{p}^{0T} - \hat{\mathbf{p}}^{(3)} \cdot \hat{\mathbf{p}}^{3T} \right) \psi = m^2 c^2 \psi \quad (48)$$

It follows that:

$$\hat{p}^{(0)} \hat{p}^{0T} - \hat{\mathbf{p}}^{(3)} \cdot \hat{\mathbf{p}}^{(3)T} = \frac{E^2}{c^2} - p^2 \quad (49)$$

so in this case:

$$p_0 = p_0^{(0)} - p_3^{(3)} \quad (50)$$

which is the classical counterpart of Eq. (34), self consistently.

For the purposes of low energy nuclear reaction (LENR) as developed in the next section, it is convenient to express Eq. (48) as:

$$(p_0^2 - m^2 c^2) \psi = p_z^2 \psi \quad (51)$$

This equation can be reduced to a Schrödinger equation using:

$$(p_0 - mc) \psi = \left( \frac{p_z^2}{p_0 + mc} \right) \psi \quad (52)$$

in which the operator is defined as:

$$p_z^2 = -\hbar^2 \nabla^2 \quad (53)$$

The results of this section are summarized as follows.

(1) A rigorously self consistent ECE theory has been developed using the classical result:

$$p_\mu^a = \left( \frac{P_0}{m^2 c^2} \right) \mathbf{p}^a \cdot \mathbf{p}^{\mu T} \quad (54)$$

which quantizes to:

$$\hat{p}_\mu^a \psi = \left( \frac{P_0}{m^2 c^2} \right) \hat{\mathbf{p}}^a \cdot \hat{\mathbf{p}}^{\mu T} \psi \quad (55)$$

(2) In the simple case of the complex circular basis superimposed on the Cartesian basis, and for motion in the Z axis:

$$(\hat{p}_0^{(0)} - \hat{p}_3^{(3)}) \psi = p_0 \psi \quad (56)$$

and

$$(\hat{p}^{(0)} \hat{p}^{0T} - \hat{\mathbf{p}}^{(3)} \cdot \hat{\mathbf{p}}^{3T}) \psi = m^2 c^2 \psi \quad (57)$$

which reduce to the Schrödinger equation.

### 3. Application to low energy nuclear reactions (LENR)

In order to explain LENR it is sufficient as in the preceding four papers UFT226 to UFT229 to use the Schrödinger limit of ECE theory. However, it is clear from the preceding section that ECE theory contains a lot more information than the Schrödinger equation. It was found in the preceding four papers that LENR can be explained straightforwardly using quantum tunnelling in the presence of resonant absorption of quanta of spacetime, typified by photons, phonons and plasmons. From Eq. (24) of UFT229 the transmission coefficient for quantum tunnelling is:

$$T = \frac{4}{\left(2\theta + \frac{1}{2\theta}\right)^2} \quad (58)$$

where:

$$\theta = \exp\left(\frac{(2\mu)^{1/2}}{\hbar} \int_a^b (V(r) - E)^{1/2} dr\right) \quad (59)$$

Here  $V$  is the barrier potential and  $E$  the kinetic energy of an incoming particle which collides with a target particle. The reduced mass of the two particles is:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad (60)$$

and LENR is considered to occur when the transmission coefficient approaches 100% or unity. It was found by numerical methods of UFT226 to UFT229 that quantum tunnelling is not alone a plausible explanation. This is obvious from the fact that LENR does not accompany ordinary chemical reactions. Quantum tunnelling must be accompanied by the resonant absorption of a quantum of spacetime, typified by the photon, phonon or plasmon. The theory of Section 2 allows this process to occur using the ECE level minimal prescription, however it can be described to a first approximation using:

$$E_1 = E_0 + \hbar\omega \quad (61)$$

In the light of UFT162 of this series ([www.aias.us](http://www.aias.us)) this procedure (first advocated by Einstein) is only a rough approximation. With this caveat it leads to:

$$\theta_1 = \exp\left(\frac{(2\mu)^{1/2}}{\hbar} \int_a^b (V(r) - E - \hbar\omega)^{1/2} dr\right) \quad (62)$$

whereupon it provides a satisfactory explanation of LENR as the numerical calculations in Section 4 show. The resonant absorption of one photon is enough to increase the transmission coefficient sufficiently.

The results of UFT162 however, point to the need for a more rigorous theory. This can be developed straightforwardly by considering the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad (63)$$

which quantizes to:

$$(E^2 - m^2 c^4)\psi = c^2 p^2 \psi \quad (64)$$

i. e.

$$(E - mc^2)\psi = \left(\frac{c^2 p^2}{E + mc^2}\right)\psi \quad (65)$$

In the rough non relativistic approximation:



$$E = \gamma mc^2 \rightarrow mc^2 \quad (66)$$

then:

$$(E - mc^2)\psi = p \frac{2}{2m} \psi \quad (67)$$

Use of the Schrödinger postulate:

$$\mathbf{p} = -i\hbar\bar{\nabla} \quad (68)$$

quantizes this equation to:

$$\frac{-\hbar^2}{2m} \bar{\nabla}^2 \psi = (E - mc^2) \psi \quad (69)$$

Add the potential energy to find:

$$\bar{\nabla}^2 \psi = \frac{-2m}{\hbar^2} (E - mc^2 - V) \psi \quad (70)$$

Now apply this with the reduced mass  $\mu$  of the fused entity in LENR to find that:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad (71)$$

where:

$$\bar{\nabla}^2 \psi = \frac{-2\mu}{\hbar^2} (E - \mu c^2 - V) \psi \quad (72)$$

$$E^2 = c^2 p^2 + \mu^2 c^4 \quad (73)$$

Accepting for the sake of argument the postulates of Einstein and de Broglie:

$$E = \hbar\omega, \quad \mathbf{p} = \hbar\boldsymbol{\kappa} \quad (74)$$

then the reduced mass is:

$$\mu = \frac{\hbar}{c^2} (\omega^2 - \boldsymbol{\kappa}^2 c^2)^{1/2} \quad (75)$$

The transmission coefficient is given by Eq. (58) with:

$$\theta = \exp\left(\frac{(2\mu)^{1/2}}{\hbar} \int_a^b (V - E + mc^2)^{1/2} dr\right) \quad (76)$$

If the fused entity is at rest then:

$$E \sim mc^2, \mu = \frac{\hbar\omega_0}{c^2} \quad (77)$$

where  $\omega_0$  is its rest angular frequency. In this case:

$$\theta \sim \exp\left(\left(\frac{2\omega_0}{\hbar}\right)^{1/2} \int_a^b \frac{V^{1/2}}{c} dr\right) \quad (78)$$

In this theory resonant absorption occurs by:

$$\omega_0 \rightarrow \omega_0 + \omega \quad (79)$$

and its effect on the transmission coefficient is worked out with Eqs. (58), (78) and (79).

In general photon, plasmon or phonon theory may be used as in the notes accompanying this paper on *www.aias.us*. Their relative efficacy in LENR is discussed in Section 4.

## 4 NUMERICAL RESULTS

Section by Drs. Doug Lindstrom and Horst Eckardt. **TO BE ADDED**

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