242(4). New Method of Determining the True Anomaly of Planar Orbits ($\theta(t)$)

Consider the equation of Lagrangian dynamics for any planar orbit:

$$\frac{d^2 r}{dt^2} + \Omega^2(r) r = 0 \quad -(1)$$

where:

$$\Omega^2 = -\frac{L_0^2}{m^2 r^4} - \frac{F(r)}{mr} \quad -(2)$$

where $L_0$ is the conserved total angular momentum, $m$ is a mass orbiting around $M$, and $F$ is the force between $m$ and $M$. The plane polar coordinate system is $(r, \theta)$.

The general solution of eq. (1) is:

$$t = \frac{1}{\sqrt{2}} \int \left( - \int r \Omega^2 dr - A \right)^{-1/2} dr + B \quad -(3)$$

where $A$ and $B$ are constants of integration.

Now consider the elliptical orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(4)$$

where $\alpha$ is the half right latitude and $\epsilon$ the eccentricity. The area of the ellipse is $\pi ab$ where

$$ab = \alpha^2 (1 - \epsilon^2)^{-3/2} \quad -(5)$$

In a time $T$ the area of the ellipse is

2) transcribed. So:

$$A(T) = \pi ab \quad -(6)$$

In a time the following area is transcribed:

$$A(t) = \frac{t}{T} A(T) \quad -(7)$$

(see Marion and Thornton, 3rd ed., chapter 7). From eq. (8) of the previous note:

$$A(T) = \frac{L_0}{2m} T \quad -(8)$$

so

$$A(t) = \frac{L_0}{2m} t \quad -(9)$$

From eq. (7) of the previous note:

$$\theta = \frac{2A}{r^2} \quad -(10)$$

so

$$\boxed{\theta(t) = \frac{L_0}{mr^2} t} \quad -(11)$$

which is consistent with:

$$\frac{d\theta}{dt} = \frac{L_0}{mr^2} \quad -(12)$$

In astronomy the quantity $\theta(t)$ is known as <u>the</u> <u>true</u> <u>anomaly</u>.

) For the elliptical orbit (4):

$$L_0^2 = \alpha m^2 \underline{M} G, \quad -(13)$$

and

$$F(r) = -\frac{m\underline{M}G}{r^2} \quad -(14)$$

so:

$$\Omega^2 = \frac{\underline{M}G}{r^3}\left(1 - \frac{\alpha}{r}\right). \quad -(15)$$

Here:

$$r_{min} = \frac{\alpha}{1+\epsilon}, \quad r_{max} = \frac{\alpha}{1-\epsilon} \quad -(16)$$

and $\epsilon$ is found from:

$$\frac{1-\epsilon}{1+\epsilon} = \frac{r_{min}}{r_{max}}. \quad -(17)$$

Here $r_{max}$ is the maximum separation of $m$ from $\underline{M}$ and $r_{min}$ the minimum separation. So $\alpha$ and $\epsilon$ may be found experimentally.

The true anomaly is therefore:

$$\boxed{\theta(t) = \frac{L_0}{\sqrt{2}\, m r^2}\left[\int\left(-\int r\Omega^2\, dr - A\right)^{-1/2} dr + B\right] \quad -(18)}$$

for any force law.

4) The perihelion is $r_{min}$, the point at which $m$ is closest to $\underline{M}$. At the perihelion:

$$\theta(\text{perihelion}) = \frac{L_0}{\sqrt{2}\, m\, r_{min}^2} \int_0^{r_{min}} \left( -\int_0^{r_{min}} r\Omega^2\, dr - A \right)^{-1/2} dr + B \qquad -(19)$$

For the ellipse, from eq. (10) of the previous note:

$$\theta(\text{perihelion}) = \frac{2\pi ab}{r_{min}^2} \qquad -(20)$$

Therefore, at the perihelion of an elliptical orbit:

$$\frac{L_0}{\sqrt{2}\, m} \int_0^{r_{min}} \left( -\int_0^{r_{min}} r\Omega^2\, dr - A \right)^{-1/2} dr + B = 2\pi ab$$

$$= 2\pi d^2 \left(1 - \epsilon^2\right)^{-3/2} \qquad -(21)$$

in which

$$\Omega^2 = \frac{\underline{M}G}{r^3}\left(1 - \frac{\alpha}{r}\right) \qquad -(22)$$

For an elliptical orbit, the angle traversed in one orbit, from perihelion to perihelion, is:

$$\theta = 2\pi \left( \frac{d^2 \left(1 - \epsilon^2\right)^{-3/2}}{r_{min}^2} \right) \qquad -(23)$$

5) Therefore

$$\theta = 2\pi (1+\epsilon)^2 (1-\epsilon^2)^{-3/2} \quad -(24)$$

and for the circle:

$$\epsilon = 0, \quad \theta = 2\pi \quad -(25)$$

The general expression (19) therefore ought to give eq. (24) when:

$$L_0^2 = \alpha m^2 MG \quad -(26)$$

$$\alpha = (1+\epsilon) r_{min} \quad -(27)$$

$$\Omega^2 = \frac{MG}{r^3}\left(1 - \frac{\alpha}{r}\right) \quad -(28)$$

$$F = -\frac{mMG}{r^2} \quad -(29)$$

i.e. eq. (21) ought to be true:

$$\frac{L_0}{\sqrt{2}m}\int_0^{r_{min}}\left(-\int_0^{r_{min}} r\Omega^2 dr - A\right)^{-1/2} dr + B$$

$$= 2\pi \alpha^2 (1-\epsilon^2)^{-3/2} \quad -(30)$$

where

$$r_{min} = \frac{\alpha}{1+\epsilon} \quad -(31)$$

This can be used as a baseline check for the calculation of $\theta$ in the Newtonian theory.

6) Initially it may be assumed that:

$$A = B = 0 \qquad -(32)$$

so:

$$\frac{L_0}{\sqrt{2}\,m}\int_0^{r_{min}}\left(-\int_0^{r_{min}}\frac{MG}{r^2}\left(1-\frac{d}{r}\right)dr\right)^{-1/2} = 2\pi d^2\left(1-\epsilon^2\right)^{-3/2}, \qquad -(33)$$

$$L_0 = m\left(d\,MG\right)^{1/2} \qquad -(34)$$

$$r_{min} = \frac{d}{1+\epsilon} \qquad -(35)$$

If this check works with the assumption (32) then the next step is to change the force law, and to recalculate eq. (18) to give $\theta$ for the new force law.

This gives the precession of the perihelion for any force law from the change in $\theta$.