242(2): Check on the Self Consistency of the Method of Note 242(1).

The initial equation is:

$$\frac{d^2 r}{dt^2} + \left(- \frac{L_0^2}{m^2 r^4} - \frac{F(r)}{mr} \right) r = 0 \quad -(1)$$

where

$$\Omega_0^2 = - \frac{L_0^2}{m^2 r^4} - \frac{F(r)}{mr} \quad -(2)$$

Eq. (1) has the solution:

$$r = r_0 \exp\left(i \Omega_0 t \right) \quad -(3)$$

$$T = \frac{2\pi}{\Omega_0} \quad -(4)$$

with

$$\Omega_0^2 = - \left(\frac{dx}{dt} + x^2 \right) \quad -(5)$$

if

$$x = i \left(\Omega_0 + \left(\frac{d\Omega_0}{dt} \right) t \right) \quad -(6)$$

Proof

Eq. (3) is of the type:

$$f = \exp\left(f_1(t) \right) \quad -(7)$$

so:

$$\frac{df}{dt} = \frac{df_1}{dt} \frac{df}{df_1} = \frac{df_1}{dt} \exp\left(f_1(t) \right) \quad -(8)$$

2) Note that Ω_0 is a function of t, so:

$$\frac{d}{dt}\exp\left(i\Omega_0(t)t\right) = \frac{d}{dt}\left(i\Omega_0(t)t\right)\exp\left(i\Omega_0(t)t\right) \qquad -(9)$$

$$= i\left(\Omega_0 + \left(\frac{d\Omega_0}{dt}\right)t\right)\exp\left(i\Omega_0(t)t\right)$$

$$= x\exp\left(i\Omega_0(t)t\right)$$

So

$$\frac{d^2}{dt^2}\exp\left(i\Omega_0(t)t\right) = \frac{dx}{dt}\exp\left(i\Omega_0(t)t\right) + x\frac{d}{dt}\left(i\Omega_0(t)t\right) \qquad -(10)$$

$$= \left(\frac{dx}{dt} + x^2\right)\exp\left(i\Omega_0(t)t\right)$$

i.e.

$$\frac{d^2r}{dt^2} - \left(\frac{dx}{dt} + x^2\right)r = 0 \qquad -(11)$$

So

$$\boxed{\Omega_0^2 = -\left(\frac{dx}{dt} + x^2\right)} \qquad -(12)$$

QED.

3) Eq. (12) is a second order differential equation for the time dependence of Ω_0, in which:

$$x = i\left(\Omega_0 + \left(\frac{d\Omega_0}{dt}\right)t\right) \quad -(13)$$

$$x^2 = -\left(\Omega_0 + \left(\frac{d\Omega_0}{dt}\right)t\right)^2 \quad -(14)$$

$$\frac{dx}{dt} = i\left(\frac{d\Omega_0}{dt} + \left(\frac{d^2\Omega_0}{dt^2}\right)t + \frac{d\Omega_0}{dt}\right)$$

$$= i\left(\left(\frac{d^2\Omega_0}{dt^2}\right)t + 2\frac{d\Omega_0}{dt}\right) \quad -(15)$$

From eq. (2) Ω_0^2 must be real valued. Therefore:

$$\left(\Omega_0 + \left(\frac{d\Omega_0}{dt}\right)t\right)^2 - i\left(\left(\frac{d^2\Omega_0}{dt^2}\right)t + 2\frac{d\Omega_0}{dt}\right)$$

$$= \Omega_0^2 \quad -(16)$$

Comparing real parts:

$$\left(\Omega_0 + \left(\frac{d\Omega_0}{dt}\right)t\right)^2 = \Omega_0^2 \quad -(17)$$

i.e. $$\left(\frac{d\Omega_0}{dt}\right)^2 t^2 + 2\left(\frac{d\Omega_0}{dt}\right)t = 0 \quad -(18)$$

4)

i.e.

$$\frac{d\Omega_0}{dt} = -\frac{2}{t} \quad -(19)$$

Eq. (17) means that it has been assumed that:

$$\left(\frac{d^2\Omega_0}{dt^2}\right)t + 2\frac{d\Omega_0}{dt} = 0 \quad -(20)$$

From eqs. (19) and (20):

$$\left(\frac{d^2\Omega_0}{dt^2}\right)t = \frac{4}{t} \quad -(21)$$

However, eqs. (19) and (21) are not self consistent, so the assumptions (17) and (20) are not valid. Therefore the complete eq. (16) must be solved by computer algebra. This may give a solution of the type:

$$\Omega_0 = \Omega_0' + i\Omega_0'' \quad -(22)$$

with real and imaginary parts. Its modulus is:

$$\Omega_0^2 = \Omega_0'^2 + \Omega_0''^2$$
$$= -\frac{L_0^2}{m^2 r^4} - \frac{F(r)}{mr} \quad -(23)$$