234(2): Minkowski Method, Conservation of Energy and Momentum.

Consider the Minkowski method applied with the infinitesimal line element:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(1)$$

$$= g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

The Lagrangian method is based on:

$$\delta \int ds = \delta \int \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right)^{1/2} d\tau = 0 \quad -(2)$$

$$= \delta \int c \, d\tau$$

and the Euler Lagrange equations:

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) = \frac{\partial \mathcal{L}}{\partial x^\mu} \quad -(3)$$

where

$$\mathcal{L} = \frac{1}{2} m c^2 \quad -(4)$$

is an invariant. Here:

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} \quad -(5)$$

From eq. (1):

$$\frac{1}{2} m c^2 = \frac{1}{2} m \left(c^2 \left(\frac{dt}{d\tau} \right)^2 - \left(\frac{dr}{d\tau} \right)^2 - r^2 \left(\frac{d\theta}{d\tau} \right)^2 \right) \quad -(6)$$

Therefore:

2)
$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(dt/d\tau)}\right) = \frac{d}{d\tau}\left(mc^2\left(\frac{dt}{d\tau}\right)\right) = \frac{\partial \mathcal{L}}{\partial} = 0 \quad -(7)$$

The quantity :
$$E = \gamma mc^2 = \left(\frac{dt}{d\tau}\right)mc^2 \quad -(8)$$

is a constant of motion.

Similarly :
$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(dr/d\tau)}\right) = \frac{d}{d\tau}\left(m\frac{dr}{d\tau}\right) = \frac{\partial \mathcal{L}}{\partial \tau} = 0, \quad -(9)$$

so the momentum component :
$$P_r = m\frac{dr}{d\tau} \quad -(10)$$

is also a constant of motion.

Similarly:
$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(d\theta/d\tau)}\right) = \frac{d}{d\tau}\left(mr^2\frac{d\theta}{d\tau}\right) = \frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad -(10)$$

so the angular momentum :
$$L = mr^2\frac{d\theta}{d\tau} \quad -(11)$$

is another constant of motion.

So there are three constants of motion and one invariant.

3) The total momentum is defined as:

$$p^2 = \gamma^2 m^2 \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right) \quad -(7)$$

$$= \gamma^2 m^2 \left(\frac{dr}{dt} \right)^2 + \frac{L^2}{r^2}$$

So p_r is the radial part of the total momentum, defined by:

$$\underline{p} = \gamma m \left(\frac{dr}{dt} \underline{e}_r + r \frac{d\theta}{dt} \underline{e}_\theta \right) \quad -(8)$$

The velocity is defined by:

$$\underline{v} = \frac{dr}{dt} \underline{e}_r + r \frac{d\theta}{dt} \underline{e}_\theta \quad -(9)$$

and

$$\underline{p} = \gamma m \underline{v}. \quad -(10)$$

The acceleration is defined by:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2) \underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \underline{e}_\theta \quad -(11)$$

$$= \frac{d}{dt} \left(\dot{r} \underline{e}_r + r \dot{\theta} \underline{e}_\theta \right)$$

The Minkowski force is defined as:

$$\underline{F}_M = d\underline{p} / d\tau \quad -(12)$$

and the Newtonian force as:

$$\underline{F}_N = d\underline{p} / dt \quad -(13)$$

4) Therefore the Newtonian force is:

$$\underline{F} = m \frac{d}{dt}\left(\gamma \underline{v}\right) \quad -(14)$$

$$= m\left(\gamma \frac{d\underline{v}}{dt} + \frac{d\gamma}{dt}\underline{v}\right)$$

where

$$\frac{d\gamma}{dt} = \frac{d\gamma}{df}\frac{df}{dv}\frac{dv}{dt} \quad -(15)$$

with

$$f = 1 - \frac{v^2}{c^2} \quad -(16)$$

so:

$$\frac{d\gamma}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{-3/2}\frac{v}{c^2}\frac{dv}{dt} \quad -(17)$$

therefore:

$$F_N = m\left(\gamma\frac{dv}{dt} + \frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\frac{dv}{dt}\right)$$

$$= m\gamma\frac{dv}{dt}\left(1 + \frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-1}\right) \quad -(18)$$

$$F_N = m\gamma^3\frac{dv}{dt} \quad -(19)$$

i.e

$$\boxed{\begin{aligned} \underline{F}_N &= m\left(1-\frac{v^2}{c^2}\right)^{-3/2}\frac{d\underline{v}}{dt}, \\ \underline{F}_m &= \gamma \underline{F}_N = m\left(1-\frac{v^2}{c^2}\right)^{-1}\frac{d\underline{v}}{dt} \end{aligned}}$$