233(7): The Description of Any Orbit in Terms of Special and General Relativity

In this note it is shown that the simplest relativistic description of any orbit is that provided by the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(1)$$

in plane polar coordinates (r, θ). From eq. (1):

$$mc^2 = mc^2 \left(\frac{dt}{d\tau}\right)^2 - m\left(\frac{dr}{d\tau}\right)^2 - mr^2\left(\frac{d\theta}{d\tau}\right)^2 \quad -(2)$$

$$= \frac{E^2}{mc^2} - \frac{p^2}{m}$$

where

$$E = mc^2\left(\frac{dt}{d\tau}\right) = \gamma mc^2 \quad -(3)$$

and

$$p^2 = m^2\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2\right)$$

$$= \gamma^2 m^2 \left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) \quad -(4)$$

$$= \gamma^2 m^2 v^2$$

So

$$\underline{p} = \gamma m \underline{v} \quad -(5)$$

and

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(6)$$

Eqs (5) and (6) are the well known equations of special relativity: the relativistic momentum and Einstein energy equation. It can be shown as follows that eq. (5) implies eq. (6). First note that eq. (1) can be written as:

2)

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(7)$$

where

$$d\underline{r} \cdot d\underline{r} = v^2 dt^2, \quad -(8)$$

so:

$$c^2 d\tau^2 = (c^2 - v^2)\, dt^2 \quad -(9)$$

and

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(10)$$

This is the well known Lorentz factor of special relativity.

From eq. (5):

$$p^2 = \gamma^2 m^2 v^2 \quad -(11)$$

so

$$p^2 c^2 = \gamma^2 m^2 c^4 \left(\frac{v^2}{c^2}\right) \quad -(12)$$

where from eq. (10):

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} \quad -(13)$$

so

$$p^2 c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right)$$

$$= \gamma^2 m^2 c^4 - m^2 c^4 \quad -(14)$$

$$= E^2 - m^2 c^4$$

so:

$$E^2 = p^2 c^2 + m^2 c^4 \quad -(15)$$

Q.E.D. Note that eq. (3) has been used in the derivation. Eq. (3) is derived from a Lagrangian analysis of eq. (2). The same Lagrangian analysis gives:

$$L = mr^2 \frac{d\theta}{d\tau} = \gamma mr^2 \frac{d\theta}{dt} \quad -(16)$$

So eq. (2) is:

$$mc^2 = \frac{E^2}{mc^2} - m\left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{mr^2} \quad -(17)$$

i.e $$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - mc^2 - \frac{L^2}{mr^2} \quad -(18)$$

where $$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\left(\frac{d\theta}{d\tau}\right)^2 = \frac{L^2}{m^2r^4}\left(\frac{dr}{d\theta}\right)^2. \quad -(19).$$

So: $$\left(\frac{dr}{d\theta}\right)^2 = \frac{mr^4}{L^2}\left(\frac{E^2 - m^2c^4}{mc^2} - \frac{L^2}{mr^2}\right)$$

$$= r^4\left(\left(\frac{p}{L}\right)^2 - \frac{1}{r^2}\right)$$

i.e $$\boxed{\left(\frac{p}{L}\right)^2 = \frac{1}{r^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)} \quad -(20)$$

<u>All orbits can be described by the ratio of p to L.</u>

This is the result of special relativity, and by Ockham's Razor is the preferred theory. Any result of a different metric must be more complicated, and therefore must be discarded by Ockham's Razor. For example consider the metric of a general spherical spacetime:

$$ds^2 = c^2 d\tau^2 = A c^2 dt^2 - B dr^2 - r^2 d\theta^2 \quad -(21)$$

as an example. It follows from eq. (21) that:

$$mc^2 = \frac{E^2}{mc^2} - \frac{p^2}{m} \quad -(22)$$

where

$$E = A^{1/2} mc^2 \frac{dt}{d\tau} \quad -(23)$$

and

$$p^2 = m^2 \left(B \left(\frac{dr}{d\tau} \right)^2 + r^2 \left(\frac{d\theta}{d\tau} \right)^2 \right) \quad -(24)$$

so

$$E^2 = m^2 c^4 + p^2 c^2 \quad -(25)$$

which has the same format as special relativity, but which contains two more parameters, A and B. Eq. (21) is:

$$ds^2 = c^2 d\tau^2 = A c^2 dt^2 - v^2 dt^2 \quad -(26)$$

where

$$v^2 = B \left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \quad -(27)$$

so

$$\gamma = \frac{dt}{d\tau} = \left(A - \frac{v^2}{c^2} \right)^{-1/2} \quad -(28)$$

which is the Lorentz factor with 1 replaced by A.

5) Therefore the metric (21) reduces to:

$$\boxed{\underline{p} = \gamma m \underline{v}} \quad -(29)$$

which means that all theories of general relativity have the same structure as special relativity. Eq. (29) is the generally covariant linear momentum. From eq. (29):

$$p^2 c^2 = \gamma^2 m^2 c^4 \left(\frac{v^2}{c^2} \right) \quad -(30)$$

where from eq. (28):

$$\frac{1}{\gamma^2} = A - \frac{v^2}{c^2} \quad -(31)$$

so

$$p^2 c^2 = A \gamma^2 m^2 c^4 - m^2 c^4 \quad -(32)$$

Using eq. (23) gives eq. (25), QED.

The orbital equation is derived by writing eq. (21) as:

$$mc^2 = \frac{E^2}{mc^2} - mB \left(\frac{dr}{d\tau} \right)^2 - \frac{L^2}{mr^2} \quad -(33)$$

so

$$mB \left(\frac{dr}{d\tau} \right)^2 = \frac{E^2}{mc^2} - mc^2 - \frac{L^2}{mr^2} \quad -(34)$$

and

$$\left(\frac{dr}{d\tau} \right)^2 = \frac{L^2}{m^2 r^4} \left(\frac{dr}{d\theta} \right)^2 \quad -(35)$$

So
$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{BL^2}\left(p^2 - \frac{L^2}{r^2}\right) \quad -(36)$$
where eq. (25) has been used.

It follows that:
$$\boxed{\left(\frac{p}{L}\right)^2 = \frac{B}{r^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)} \quad -(37)$$

In special relativity:
$$B = 1. \quad -(38)$$

Therefore:
$$\boxed{\left(\frac{p}{L}\right)^2_{GR} = B\left(\frac{p}{L}\right)^2_{SR}} \quad -(39)$$

The only effect of general relativity is to make the analysis more complicated by introducing the parameter B. Therefore general relativity of this type is discarded by Ockham's Razor of natural philosophy.

The observable orbit of any type anywhere in the universe is $dr / d\theta$. Its simplest relativistic description is special relativity. Any other description is superfluous.

7) The Minkowski metric is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(40)$$

This can be factorized into Cartan tetrads:

$$g_{\mu\nu} = q^a_\mu \, q^b_\nu \, \eta_{ab} \quad -(41)$$

where

$$\eta_{ab} = \eta^{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(42)$$

in the same Cartesian basis. In the complex circular basis the structure of η_{ab} is different. The tetrads q^a_μ and q^b_ν are phase dependent and as in previous work give torsion and curvature. The Cartan and Evans identities then give the field equations of gravitation.

This is a simpler and more complete theory than Einsteinian general relativity, able to describe all orbits. EGR omits torsion and in fact is unable to describe any orbit.
