233(6): Thomas Precession and x Factor.

Consider the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(1)$$

in a plane, with polar coordinates (r, θ). Here τ is the proper time and t the observer time. By definition:

$$ds^2 = c^2 dt^2 - \underline{dr} \cdot \underline{dr} \quad -(2)$$

where

$$\underline{dr} \cdot \underline{dr} = dr^2 + r^2 d\theta^2 = v^2 dt^2 \quad -(3)$$

where v is the velocity in the observer frame. So:

$$c^2 d\tau^2 = (c^2 - v^2) dt^2 \quad -(4)$$

and

$$d\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt. \quad -(5)$$

Therefore

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(6)$$

is the Lorentz factor.

A Lagrangian analysis of eq. (1) as in previous work shows that:

$$E = \gamma m c^2, \quad L = \gamma m r^2 \frac{d\theta}{dt}, \quad -(7)$$

$$p = \gamma m v.$$

It may also be shown that:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4 \left(\left(\frac{p}{L}\right)^2 - \frac{1}{r^2} \right) \quad -(8)$$

2)

i.e. $$\left(\frac{dr}{d\theta}\right)^2 = \frac{v^2}{\omega^2} - r^2 \quad -(9)$$

where $$\omega = \frac{d\theta}{dt} \quad -(10)$$

therefore $$\boxed{v^2 = \omega^2\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)} \quad -(11)$$

From eq. (3): $$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\omega^2 \quad -(12)$$

Eqs. (11) and (12) are the same because:

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \omega\frac{dr}{d\theta} \quad -(13)$$

For the Newtonian ellipse:

$$\frac{dr}{d\theta} = \frac{\epsilon}{\alpha} r^2 \sin\theta \quad -(14)$$

where ϵ is the ellipticity and α the half right latitude. For the precessing ellipse:

$$\frac{dr}{d\theta} = \frac{x\epsilon}{\alpha} r^2 \sin(x\theta) \quad -(15)$$

where x is the precession constant.

The difference between eqs. (14) and (15) may be expressed as:

3)

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha}\left(x\sin(x\theta) - \sin\theta\right), \quad -(16)$$

and if:

$$x \sim 1, \quad -(17)$$

then:

$$\sin(x\theta) \sim \sin\theta. \quad -(18)$$

In the solar system this is an excellent approximation. So

$$\frac{dr}{d\theta} \sim x\frac{\epsilon r^2}{\alpha}\sin\theta. \quad -(19)$$

i.e

$$r \sim x\left(\frac{\alpha}{1+\epsilon\sin\theta}\right). \quad -(20)$$

Denote this result as:

$$r \sim x\, r_0. \quad -(21)$$

For an approximately circular orbit, the circumference is $2\pi r$, so in a rotation of 2π:

$$2\pi r \rightarrow 2\pi r(x-1). \quad -(22)$$

The same result can be seen by considering that after a rotation of 2π,

$$\theta = 2\pi \quad -(23)$$

$$x\theta = 2\pi x \quad -(24)$$

so

$$\boxed{\Delta\theta = 2\pi(x-1)} \quad -(25)$$

As in UFT110, the precession is described by

4) changing:

$$\theta \to \theta + \Omega t \quad -(26)$$

in eq. (1), so:

$$\theta' = \theta + \Omega t \quad -(27)$$

and

$$d\theta' = d\theta + \Omega dt. \quad -(28)$$

It follows that:

$$ds'^2 = (c^2 - r^2\Omega^2)\,dt^2 + 2\Omega r^2\, d\theta\, dt - dr^2 - r^2 d\theta^2 \quad -(29)$$

$$= c^2 d\tau'^2$$

So:

$$d\tau'^2 = \left(1 - \left(\frac{r\Omega}{c}\right)^2\right)\left(dt^2 + \frac{2\Omega r^2\, d\theta\, dt}{c^2\left(1 - \left(\frac{r\Omega}{c}\right)^2\right)}\right) - \frac{dr^2 + r^2 d\theta^2}{c^2} \quad -(30)$$

In eq. (30):

$$d\tau^2 := \left(1 - \left(\frac{r\Omega}{c}\right)^2\right) dt^2 \quad -(31)$$

and

$$d\Omega_1 := \frac{d\Omega}{1 - \left(\frac{r\Omega}{c}\right)^2} \quad -(32)$$

and the rotation (26) is considered to introduce the phase difference:

$$\Delta\phi = \Omega_1 \tau - \Omega t \quad -(33)$$

5) In a rotation of 2π:

$$\Delta \phi = 2\pi \left(x - 1 \right) \quad -(34)$$

where

$$x = \left(1 - \left(\frac{r\Omega}{c} \right)^2 \right)^{1/2} \quad -(35)$$

So there is a relation between Thomas precession and the precession of an ellipse.

The relation between the Minkowski metric and the ellipse is revealed by writing eq. (8) as:

$$\left(\frac{d\theta}{dr} \right)^2 = \frac{1}{r^4} \cdot \frac{1}{\left(\frac{p}{L}\right)^2 - \frac{1}{r^2}} \quad -(36)$$

$$= \frac{L^2}{r^2} \left(\frac{1}{r^2 p^2 - L^2} \right)$$

In Newtonian dynamics:

$$\left(\frac{d\theta}{dr} \right)^2 = \frac{L^2}{r^4} \left(\frac{1}{2m\left(E - U - \frac{L^2}{2mr^2}\right)} \right) \quad -(37)$$

$$= \frac{L^2}{r^2} \left(\frac{1}{2mr^2(E-U) - L^2} \right)$$

Eqs (36) and (37) are the same if:

$$r^2 p^2 = 2mr^2 \left(E - U \right) \quad -(38)$$

i.e. if:

6)

$$\frac{p^2}{2m} = E - U \quad -(39)$$

or

$$E = \frac{p^2}{2m} + U \quad -(40)$$

The ellipse is obtained from Newtonian dynamics by:

$$U = -\frac{k}{r} \quad -(41)$$

as are all conical sections as first shown by Bernoulli.

<u>However</u>, <u>without any reference at all to Newtonian dynamics</u> we may write:

$$\left(\frac{d\theta}{dr}\right)^2 = \frac{L^2}{r^2}\left(\frac{1}{r^2p^2 - L^2}\right) = \frac{\alpha^2}{\epsilon^2 r^4 \sin^2\theta} \quad -(42)$$

so:

$$\left(\frac{dr}{d\theta}\right)^2 = r^2\left(\left(\frac{rp}{L}\right)^2 - 1\right) = \frac{\epsilon^2 r^4 \sin^2\theta}{\alpha^2} \quad -(43)$$

and so:

7)

$$\left(\frac{rp}{L}\right)^2 - 1 = \left(\frac{\epsilon r \sin\theta}{\alpha}\right)^2 \quad -(44)$$

or

$$\boxed{\left(\frac{p}{L}\right)^2 = \frac{1}{r^2} + \frac{\epsilon^2 \sin^2\theta}{\alpha^2}} \quad -(45)$$

The elliptical orbit is a ratio of p/L determined by eq. (45). Here:

$$p = \gamma m v \quad -(46)$$

$$L = \gamma m r^2 \frac{d\theta}{dt} = \gamma m r^2 \omega .$$

Note that eq. (45) is a correctly relativistic theory of the ellipse. The Newtonian theory is not correctly relativistic.

The precessing ellipse is described by:

$$\boxed{\left(\frac{p}{L}\right)^2 = \frac{1}{r^2} + \frac{x^2 \epsilon^2 \sin^2(x\theta)}{\alpha^2}} \quad -(47)$$

where x is given by the Thomas precession in eq. (35).

8.) For the precessing ellipse:

$$\sin^2(x\theta) = 1 - \frac{1}{\epsilon^2}\left(\frac{d}{r} - 1\right)^2 \quad -(48)$$

$$= 1 - \frac{1}{\epsilon^2}\left(\frac{d-r}{r}\right)^2 = \frac{\epsilon^2 r^2 - (d-r)^2}{\epsilon^2 r^2}$$

So:

$$\left(\frac{p}{L}\right)^2 = \frac{1}{r^2} + \left(\frac{x\epsilon}{d}\right)^2\left(\frac{\epsilon^2 r^2 - (d-r)^2}{\epsilon^2 r^2}\right) \quad -(49)$$

$$= \frac{1}{r^2}\left(1 + \left(\frac{x}{d}\right)^2\left(\epsilon^2 r^2 - (d-r)^2\right)\right)$$

As in note 233(3):

$$\left(\frac{pr}{L}\right)^2 = 1 + x^2\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{(r_{max} - r)(r - r_{min})}{r_{max}^2}$$

$$= 1 + x^2\left(\frac{1-\epsilon}{1+\epsilon}\right)\frac{(r_{max} - r)(r - r_{min})}{r_{min}^2}$$

$$-(50)$$

It is clear that this is a fully relativistic result that replaces both Einsteinian and Newtonian dynamics.