233(5): Recalculation of Note 233(2) with a θ dependent x.

In general the orbit is given by:

$$r = \frac{\alpha}{1+\epsilon\cos(\theta x(\theta))}, \quad -(1)$$

so

$$\frac{dr}{d\theta} = -\frac{d}{d\theta}\left(\cos(\theta x(\theta))\right)\frac{\epsilon r^2}{\alpha}. \quad -(2)$$

Denote

$$f = \cos u, \quad u = \theta x(\theta) \quad -(3)$$

and

$$\frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta}. \quad -(4)$$

So

$$\frac{d}{d\theta}\left(\cos(\theta x(\theta))\right) = -\frac{d}{d\theta}\left(\theta x(\theta)\right)\sin(\theta x(\theta)) \quad -(5)$$

$$= -\left(x(\theta) + \theta\frac{dx}{d\theta}\right)\sin(\theta x(\theta)).$$

So:

$$\frac{dr}{d\theta} = \left(x(\theta) + \theta\frac{dx}{d\theta}\right)\frac{\epsilon}{\alpha} r^2 \sin(\theta x(\theta)). \quad -(6)$$

Denote

$$y = x + \theta\frac{dx}{d\theta} \quad -(7)$$

then as in note 233(5), the quantity y can be determined from:

2)

$$\left(\frac{v}{r\omega}\right)^2 = 1 + y^2\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{(r_{max}-r)(r-r_{min})}{r_{max}^2} \quad -(8)$$

$$= 1 + y^2\left(\frac{1-\epsilon}{1+\epsilon}\right)\frac{(r_{max}-r)(r-r_{min})}{r_{min}^2}$$

So x of note 233(2) is replaced by y.
If x is assumed to be a constant then

$$y = x. \quad -(9)$$

As shown in previous work, any orbit can be described with the precessing conical section, i.e..

$$r = f(\theta) = \frac{\alpha}{1+\epsilon\cos(\theta x(\theta))} \quad -(10)$$

and

$$\frac{df(\theta)}{d\theta} = \frac{y\epsilon r^2}{\alpha}\sin(\theta x(\theta)). \quad -(11)$$

$$= y\,\epsilon\alpha\,\frac{\sin(\theta x(\theta))}{1+\epsilon\cos(\theta x(\theta))}$$

Therefore any orbit may be synthesized using:

$$f(\theta) = \epsilon\alpha\int\left(x+\theta\frac{dx}{d\theta}\right)\frac{\sin(\theta x(\theta))}{1+\epsilon\cos(\theta x(\theta))}\,d\theta \quad -(12)$$

3)

i.e.:

$$f(\theta) = \epsilon\alpha\left(\int \frac{x(\theta)\sin(\theta x(\theta))\,d\theta}{1+\epsilon\cos(\theta x(\theta))} + \int \frac{\theta\sin(\theta x(\theta))\,dx}{1+\epsilon\cos(\theta x(\theta))}\right) \quad -(13)$$

If x is assumed to be a constant then:

$$f(\theta) = \epsilon\alpha x \int \frac{\sin(\theta x)\,d\theta}{1+\epsilon\cos(\theta x)} \quad -(14)$$

Let $$\phi = x\theta \quad -(15)$$

then $$d\theta = \frac{1}{x}\,d\phi \quad -(16)$$

and $$f(\theta) = \epsilon\alpha \int \frac{\sin\phi\,d\phi}{1+\epsilon\cos\phi} \quad -(17)$$

Computer algebra may now be used to evaluate the integrals (13) and (17).