233(4): True Precessing Ellipse from the Cartan Metric.

The true precessing ellipse is:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(1)$$

where:

$$\alpha = (1+\epsilon)\, r_{min} = (1-\epsilon)\, r_{max}. \quad -(2)$$

Here $\alpha$ is the half right latitude, $r_{min}$ and $r_{max}$ the distance of closest approach and furthest separation, $\epsilon$ the ellipticity, $x$ the precession factor and $(r, \theta)$ the plane polar coordinate system. From eq. (1):

$$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{\epsilon x}{\alpha}\right)^2 r^4 \sin^2(x\theta) \quad -(3)$$

$$= \left(\frac{\epsilon x}{\alpha}\right)^2 r^4 \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)$$

$$= x^2 \left(\frac{r}{\alpha}\right)^2 \left(\epsilon^2 r^2 - (\alpha - r)^2\right)$$

$$= x^2 r^2 \left(\frac{1-\epsilon}{1+\epsilon}\right)\left(\frac{(r_{max}-r)(r-r_{min})}{r_{min}^2}\right)$$

$$= x^2 r^2 \left(\frac{1+\epsilon}{1-\epsilon}\right)\left(\frac{(r_{max}-r)(r-r_{min})}{r_{max}^2}\right)$$

In the Newtonian limit:

$$x = 1 \quad -(4)$$

2) As in UFT195, the Crothers metric is:

$$ds^2 = c^2 d\tau^2 = AC^{1/2} c^2 dt^2 - BC^{1/2} dr^2 - C(r) d\theta^2 \quad -(5)$$

where

$$C(r) = \left( |r - r_0|^n + \alpha_1^n \right)^{2/n} \quad -(6)$$

This gives:

$$E = mc^2 A C^{1/2}(r) \left( \frac{dt}{d\tau} \right) \quad -(7)$$

$$L = m C(r) \frac{d\theta}{d\tau}, \quad -(8)$$

$$\frac{dt}{d\tau} = \left( A C^{1/2}(r) - \frac{v^2}{c^2} \right)^{-1/2} \quad -(9)$$

and

$$\left( \frac{dr}{d\theta} \right)^2 = \frac{m C(r)}{B L^2} \left( \frac{E^2}{A m c^2} - C^{1/2}(r) \left( mc^2 + \frac{L^2}{m C(r)} \right) \right) \quad -(10)$$

$$= x^2 r^2 \left( \frac{1+\epsilon}{1-\epsilon} \right) \left( \frac{(r_{max} - r)(r - r_{min})}{r_{max}^2} \right)$$

$$= x^2 r^2 \left( \frac{1-\epsilon}{1+\epsilon} \right) \left( \frac{(r_{max} - r)(r - r_{min})}{r_{min}^2} \right)$$

The quantities on the right hand side of eq. (10) are all experimentally measurable.

3) The Crothers metric can be interpreted as a perturbation of the Minkowski metric:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad - (5)$$

and reduces to the Minkowski metric when:

$$AC^{1/2} = BC^{1/2} = 1, \quad C(r) = r^2; \quad - (6)$$

i.e

$$A = B = \frac{1}{r}, \quad - (7)$$

so eq. (10) becomes:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4 \left( \frac{E^2 - m^2 c^4}{c^2 L^2} - \frac{1}{r^2} \right) \quad - (7)$$

$$= r^4 \left( \left(\frac{p}{L}\right)^2 - \frac{1}{r^2} \right)$$

as in note 233(3).

In this limit:

$$E = \gamma m c^2, \quad p = \gamma m v \quad - (8)$$

so

$$\left(\frac{dr}{d\theta}\right)^2 = r^2 \left( \left(\frac{v}{r\omega}\right)^2 - 1 \right). \quad - (10)$$

For a circular orbit:

$$\frac{dr}{d\theta} = 0. \quad - (11)$$

The above procedure assumes that:

$$r_0 = 0, \quad \alpha_1 = 0. \quad - (12)$$

4) However, the presence of a finite $r_0$ is important to eliminate unphysical singularities. The original Schwarzschild metric contained a finite $r_0$. When this is properly considered there is no big bang and no black holes. So as a result approximation it is reasonable to assume:

$$A = B = \frac{1}{|r - r_0|}, \quad C(r) = |r - r_0|^2 \qquad -(13)$$

i.e.

$$n = 1, \quad \alpha_1 = 0 \qquad -(14)$$

in eq. (6). This means that the whole of the calculation is carried through with $r$ replaced by $|r - r_0|$.

In the Newtonian limit this procedure leads to the result:

$$\left(\frac{v}{|r - r_0|\omega}\right)^2 = 1 + \left(\frac{1+\epsilon}{1-\epsilon}\right)\left(\frac{(r_{max} - |r - r_0|)(|r - r_0| - r_{min})}{r_{max}^2}\right) \qquad -(15)$$

and that the true precessing ellipse is:

$$\boxed{|r - r_0| = \frac{\alpha}{1 + \epsilon\cos(x\theta)}} \qquad -(16)$$

5) In the Crothers metric:

$$r_0 = \frac{2MG}{c^2} \quad -(17)$$

as given originally by Schwarzschild in his letter to Einstein of 22rd Dec. 1915.

This is a simple illustration of the use of the Schwarzschild/Crothers theory. In this case the distance between object m and object M is measured by $|r - r_0|$.

The false Schwarzschild metric is:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_0}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_0}{r}\right)} - r^2 d\theta^2 \quad -(18)$$

and there is a singularity at $r = 0$. $-(19)$

but a simple example of the true Schwarzschild/Crothers metric is:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_0}{|r - r_0|}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_0}{|r - r_0|}\right)} - |r - r_0|^2 d\theta^2 \quad -(19)$$

and there is no singularity at $r = 0$.