233(1) Use of the Cartan Metric to Produce a Precessing Ellipse, Review of the Conventional "Schwarzschild" Method.

This method starts with the arbitrary assertion that:

$$ds^2 = c^2 d\tau^2 = \left(1-\frac{r_0}{r}\right)c^2 dt^2 - \left(1-\frac{r_0}{r}\right)^{-1} dr^2 - r^2 d\theta^2 \quad -(1)$$

This infinitesimal line element is misattributed to K. Schwarzschild. In historical fact the latter wrote a letter to A. Einstein on 22nd Dec. 1915 dismissing the Einstein theory and introducing ds^2 completely different from (1). For the sake of review, the conventional method is to proceed as follows:

$$mc^2 = mc^2\left(1-\frac{r_0}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - m\left(1-\frac{r_0}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - mr^2\left(\frac{d\theta}{d\tau}\right)^2 \quad -(2)$$

from which:

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}mc^2\left(\left(1-\frac{r_0}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{r_0}{r}\right)\right) - \frac{mr^2}{2}\left(1-\frac{r_0}{r}\right)\left(\frac{d\theta}{d\tau}\right)^2 \quad -(3)$$

The Lagrangian method is used to define two constants of motion:

$$E = mc^2\left(1-\frac{r_0}{r}\right)\frac{dt}{d\tau} \quad -(4)$$

$$L = mr^2\frac{d\theta}{d\tau} \quad -(5)$$

Here τ is the proper time, t the time in the observer

) frame, (r, θ) the plane polar coordinate system, and:

$$r_0 = \frac{2MG}{c^2} \quad -(6)$$

where M is the gravitating mass, G the Newton constant, c the speed of light in a vacuum. The mass m is attracted to M by gravitation.

Eq. (3) can be written as:

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{mc^2 r_0}{2r} + \frac{L^2}{2mr^2}\left(1 - \frac{r_0}{r}\right) \quad -(7)$$

$$= \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{2mr^2} - \frac{mMG}{r} - \frac{L^2MG}{mc^2r^3}.$$

Although this method is purely geometrical, an "effective potential" is introduced:

$$V = \frac{L^2}{2mr^2} - \frac{mMG}{r} - \frac{L^2MG}{mc^2r^3} \quad -(8)$$

<u>This</u> is <u>a completely arbitrary procedure</u>, because it mixes up concepts. The potential (8) is a Newtonian concept, while eq. (1) is a concept of geometry. In eq. (8), the centripetal force is asserted arbitrarily to be the positive or repulsive term. The first negative term is said to be the Hooke / Newton law of attraction. The

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Second negative term is said to be a correction due to general relativity.

The use of Newtonian methods continues by defining the force of attraction as:

$$F = -\frac{\partial V}{\partial r} = -\frac{mMG}{r^2} - \frac{3L^2MG}{mc^2r^4} \quad -(9)$$

This is the usual Hooke Newton force plus a correction inversely proportional to the fourth power of r. Finally this force is used in a classical Lagrange method that gives the equation:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{m}{L^2} r^2 F \quad -(10)$$

$$= \frac{Gm^2M}{L^2} + \left(\frac{3MG}{c^2}\right)\frac{1}{r^2}$$

As in UFT 232:

$$\frac{1}{\alpha} := \frac{Gm^2M}{L^2} \quad ; \quad \delta := \frac{3MG}{c^2} \quad -(11)$$

So

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} - \frac{1}{\alpha} - \frac{\delta}{r^2} = 0 \quad -(12)$$

It was found that eq. (12) does not give the true precessing ellipse.

4) This is exactly the criticism made by Schwarzschild on Dec 22rd. 1915, but not using the contemporary language of this note. There are many problems with this traditional method, the most glaring one is the use of classical dynamics in a relativistic context.

These problems are best illustrated by going back to basics and considering the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(13)$$

$$= c^2 dt^2 - v^2 dt^2$$

so:

$$mc^2 = \frac{E^2}{mc^2} - \frac{p^2}{m} \quad -(14)$$

which is the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(15)$$

with:

$$E = mc^2 \frac{dt}{d\tau} = \gamma mc^2 \quad -(16)$$

$$p^2 = m^2 \left(\left(\frac{dr}{d\tau} \right)^2 + r^2 \left(\frac{d\theta}{d\tau} \right)^2 \right) \quad -(17)$$

$$= \gamma^2 m^2 v^2$$

where:

$$v^2 = \left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 . \quad -(18)$$

5) Therefore

$$\underline{p} = \gamma m \underline{v} \quad -(19)$$

is the relativistic momentum of special relativity and

$$E = \gamma m c^2 \quad -(20)$$

is the relativistic energy. From previous work in the UFT series it is seen that the Minkowski metric can produce an orbit. So EGR is not needed at all. Setting this point aside for the sake of argument, eq. (2) can be written as:

$$mc^2 = mc^2\left(1-\frac{r_0}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - m\left(\left(\frac{dr}{d\tau}\right)^2\left(1-\frac{r_0}{r}\right)^{-1} + r^2\left(\frac{d\theta}{d\tau}\right)^2\right) \quad -(21)$$

so:

$$\left(1-\frac{r_0}{r}\right)mc^2 = mc^2\left(1-\frac{r_0}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - m\left(\left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{r_0}{r}\right)r^2\left(\frac{d\theta}{d\tau}\right)^2\right)$$

$$= \frac{E^2}{mc^2} - \frac{p^2}{m} \quad -(22)$$

where

$$E = \left(1-\frac{r_0}{r}\right)mc^2\,\frac{dt}{d\tau} \quad -(23)$$

$$p^2 = m^2\left(\left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{r_0}{r}\right)r^2\left(\frac{d\theta}{d\tau}\right)^2\right) \quad -(24)$$

If:

$$r_0 \ll r \quad -(25)$$

b) then EGR is a small perturbation of special relativity. Furthermore it is a perturbation of special relativity without a potential energy.

Eq. (22) can be written as:

$$\boxed{E_1^2 = c^2 p_1^2 + m^2 c^4} \quad -(26)$$

where:

$$E_1 = \left(1 - \frac{r_0}{r}\right)^{1/2} mc^2 \frac{dt}{d\tau}, \quad -(27)$$

$$p_1 = \left(1 - \frac{r_0}{r}\right)^{-1/2} p, \quad -(28)$$

i.e.

$$mc^2 = \frac{E^2}{mc^2}\left(1 - \frac{r_0}{r}\right)^{-1} - \frac{p^2}{m}\left(1 - \frac{r_0}{r}\right)^{-1}$$

$$= \frac{E_1^2}{mc^2} - \frac{p_1^2}{m}. \quad -(29)$$

From eq. (1):

$$ds^2 = c^2 dt^2\left(1 - \frac{r_0}{r}\right) - dr^2\left(1 - \frac{r_0}{r}\right)^{-1} - r^2 d\theta^2 \quad -(30)$$

$$\sim c^2 dt^2\left(1 - \frac{r_0}{r}\right) - dr^2\left(1 + \frac{r_0}{r}\right) - r^2 d\theta^2$$

for

$$r_0 \ll r. \quad -(31)$$

Therefore to an excellent approximation:

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$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - \frac{r_0}{r}\left(c^2 dt^2 + dr^2\right) \quad -(32)$$

So:

$$mc^2 = \frac{E^2}{mc^2} - \frac{p^2}{m} - m\frac{r_0}{r}\left(c^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2\right), \quad -(33)$$

i.e.

$$E^2 = c^2 p^2 + m^2 c^4 + m^2 c^2 \frac{r_0}{r}\left(c^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2\right) \quad -(34)$$

By definition:

$$v = \frac{dr}{dt} \quad -(35)$$

so:

$$E^2 = c^2 p^2 + m^2 c^4 + m^2 c^2 \frac{r_0}{r}\left(\frac{dt}{d\tau}\right)^2 \left(c^2 + v^2\right) \quad -(35)$$

$$= c^2 p^2 + m^2 c^2\left(c^2 + \frac{r_0}{r}\left(\frac{dt}{d\tau}\right)^2 \left(c^2 + v^2\right)\right)$$

i.e

$$\boxed{E^2 = c^2 p^2 + m^2 c^2\left(c^2\left(1 + \frac{r_0}{r}\left(\frac{dt}{d\tau}\right)^2\right) + \frac{r_0}{r}\left(\frac{dt}{d\tau}\right)^2 v^2\right)} \quad -(36)$$

From eq. (1):

$$c^2 d\tau^2 = c^2 dt^2\left(1 - \frac{r_0}{r}\right) - v^2 dt^2 \quad -(37)$$

8) So:

$$\left(\frac{dt}{d\tau}\right)^2 = \left(1 - \frac{r_0}{r} - \frac{v^2}{c^2}\right)^{-1} \quad -(38)$$

In general relativity of the Einstein type the rest energy

$$E_0 = mc^2 \quad -(39)$$

is regarded as an invariant. So eq. (35) is most clearly written as:

$$E^2 = c^2\left(p^2 + m^2 \frac{r_0}{r}\left(\frac{dt}{d\tau}\right)^2 (c^2+v^2)\right) + m^2c^4 \quad -(40)$$

Therefore there is an additional:

$$\boxed{\pi^2 := m^2 \frac{r_0}{r}\left(1 - \frac{r_0}{r} - \frac{v^2}{c^2}\right)^{-1}(c^2+v^2)} \quad -(41)$$

and

$$E^2 = c^2(p^2 + \pi^2) + m^2c^4. \quad -(42)$$

If

$$v \ll c \quad -(43)$$

as in planetary motion then:

$$\pi^2 \sim m^2 \frac{r_0}{r}\left(1 - \frac{r_0}{r}\right)^{-1} c^2 \quad -(44)$$

It has been assumed that:

$$r_0 \ll r \quad -(45)$$

So:

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$$\pi^2 \sim m^2 \frac{r_0}{r} c^2 = 2m^2 \frac{MG}{r} \quad -(46)$$

So an additional velocity is added to the orbit:

$$v \doteq \left(\frac{2MG}{r}\right)^{1/2}. \quad -(47)$$

For earth:

$$G = 6.674 \times 10^{-11}\ Nm^2 kg^{-2}$$
$$M\ (\text{sun}) = 1.989 \times 10^{30}\ kg$$
$$r\ (\text{mean}) = 1.496 \times 10^{12}\ m$$

So

$$v = 1.332 \times 10^{4}\ ms^{-1} \quad -(48)$$

and the Einstein energy equation is modified in this approximation to:

$$E^2 = c^2 p^2 + m^2 c^4 + 2m^2 \frac{MG}{r} c^2 \quad -(49)$$

Therefore:

$$E^2 = c^2 p^2 + m^2 c^2 \left(c^2 + \frac{2MG}{r}\right) \quad -(50)$$

The classical gravitational potential is:

$$V = -\frac{MG}{r} \quad -(51)$$

From eqs. (49) and (51):

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$$\frac{2m^2 mGc^2}{r} = -2m^2c^2 V \quad -(52)$$

$$= E^2 - (c^2p^2 + m^2c^4)$$

So

$$V = -\frac{\left(E^2 - (c^2p^2 + m^2c^4)\right)}{2m^2c^2} \quad -(53).$$

This is the actual result of EGR in the given approximations, and it is different from the arbitrary procedure represented by eq. (7).